JEE ADVANCED PAPER 1 :CODE 2

PART III : MATHEMETICS

SECTION – 1 : (One or More Than One Options Correct Type)

Q41. Let M be a 2 x 2 symmetric matrix with integer entries. Then M is invertible if

(A)the first column of M is the transpose of the second row of M
(B)the second row of M is the transpose of the first column of M
(C)M is a diagonal matrix with nonzero entries in the main diagonal
(D)the product of entries in the main diagonal of M is not the square of an Integer

Sol.
$M = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$
$|M| = ab - c^2$
$\begin{bmatrix} a \\ c \end{bmatrix} = [c\, b]^T = \begin{bmatrix} c \\ b \end{bmatrix}$
$\Rightarrow a = b = c$
$|M| = 0 \Rightarrow$ M is not invertible
(b)
$[c\ b] = \begin{bmatrix} a \\ c \end{bmatrix}^T = [a\ c]$
$\Rightarrow a = b = c$ M is not invertible
(c)
$|M| = ab - 0 = ab \neq 0$
$\therefore$ M is invertible
(d)
$|M| = ab - c^2 \neq 0$
$\therefore$ M is invertible

Q42. A circle S passes through the point (0,1) and is orthogonal to the circles $(x-1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then
(A)radius of S is 8
(B)radius of S is 7
(C)centre of S is (-7,1)
(D)centre of S is (-8,1)

Sol. let equation of circle be
$x^2 + y^2 + 2gx + 2fy + c = 0$
It passes though (0, 1)
$0 + 1 + 2g(0) + 2f + c = 0$ ......(1)
$(x-1)^2 + y^2 = 16 \Rightarrow x^2 + y^2 - 2x - 15 = 0$ $\qquad g_1 = -1, c_1 = -15$
$X^2 + y^2 = 1 \quad \Rightarrow x^2 + y^2 = 1$ $\qquad g_2 = 0, C_2 = -1$

Orthogonality condition $2g_1 g_2 + 2f_1 f_2 = c_1 + c_2$

$2g(-1) + 2f(0) = -15 + c$

$2g(0) + 2f(0) = c - 1$

$(c = 1)$ - (3)

from (1), 2 and (3) $g = 7$ $f = -1$

so. $x^2 + y^2 + 14x - 2y + 1 = 0$

$(x + 7)^2 + (y - 1)^2 = 49$

{center = (-7, 1), radius = 7}

Q 43. Let x, y and z be three vectors each of magnitude √2 and the angle between each pair of them is π/3. If a is a nonzero vector perpendicular to x and y x z and b is nonzero vector perpendicular to y and z x x, then

(A) b = (b . z) (z - x)

(B) a = (a . y) (y - z)

(C) a . b = (a . y) (b . z)

(D) a = (a . y) (z - y)

Sol. (A) b = $((\vec{b}.\vec{z})(\vec{z} - \vec{x})$

Multiplying by z

$(\vec{b} . \vec{z}) = (\vec{b} . \vec{z})(\vec{z}.\vec{z}) - (\vec{x}.\vec{z})$

$1 = 2 - \sqrt{2} - \sqrt{2} \times \frac{1}{2}$

Option A is correct

(B) $\vec{a} = (\vec{a}\vec{y})(\vec{y} - \vec{z})$

Multiplying by $\vec{y}$

$(\vec{a}\vec{y}) = (\vec{a}\vec{y})(\vec{y}^2 - \vec{y}\vec{z})$

$1 = 2 - \sqrt{2}\sqrt{2 \cdot \frac{1}{2}}$

So Option (B) is correct

Q44. From a point P (λ, λ, λ), perpendiculars PQ and PR are drawn respectively o the lines y = x, z = 1 and y = -x, z = - 1. If P is such that ∠QPR is a right angle, then the possible value(s) of λ is(are)

(A) √2

(B) 1

(C) -1

(D) -√2

Sol.

(B,C)

$L_1 \frac{x}{1} = \frac{y}{1} = \frac{z-1}{0}$ Q $(t_1, t_1, 1)$

$L_2 \frac{x}{1} = \frac{y}{-1} = \frac{z+1}{0}$ R $(t_2, -t_2, -1)$

dv or PQ = ($\lambda - t_1$, $\lambda - t_1$, λ -1) = dv or l_2
dv or PR = ($\lambda - t_2$, $\lambda + t_2$,λ + 1) = dv or l_1
$\lambda - 1 = 0$ or $\lambda + 1 = 0 \Rightarrow \lambda = 1$ or $\lambda = -1$

Q45. For every pair of continuous functions f, g: [0,1] $\to\mathbb{R}$ such thatmax {f (x): x $\in$ [0,1]} = max {g (x): x $\in$ [0,1]),the correct statement(s) is(are) :
(A)$(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some c $\in$[0,1]
(B)$(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some c $\in$[0,1]
(C)$(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some c $\in$ [0,1]
(D)$(f(c))^2 = (g(c))^2$ for some c $\in$ [0,1]

Sol. Sol. max $\{f\ x\ : x \epsilon [0,1]\}$ = max { g (x) = x ϵ [0.1]}
For same c ϵ [0.1]
F(c) = g (c)
$\Rightarrow [f(c)]^2 = [g(c)]^2$
So, option (A) and (D) are correct

Q 46 Let M and N be two 3 x 3 matrices such that MN = NM. Further, if $M \neq N^2$ and $M^2 = N^4$, then
(a) Determinant of $(M^2 + MN^2)$ is 0
(b) There is a 3 x3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix
(c) Determinant of $(M^2 + MN^2) \geq 1$
(d) For a 3x3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix

Sol. MN = NM $\Rightarrow$ M and N are Aymmethic
$|M^2 + MN^2| = |M|\ |M + N^2|$

Q 47.
Let a $\in\mathbb{R}$ and let $f:\mathbb{R}\to\mathbb{R}$ be given by
f (x) = x^5 — 5x + a.
Then
(a) f (x) has only one real root if a > 4
(b) f (x) has only one real root if a < 4
(c) f (x) has three real roots if a < —4
(d) f (x) has three real roots if —4 < a < 4

Sol. F (x) = $x^5 - 5x + a$
$f'(x) = 5x^4 - 5. \Rightarrow f'(x) = 0$ at $x = \pm 1$ f(0) = a
$f(-\infty) = -\infty$ f (1) = - 4 + a
$f(\infty) = \infty$ f (-1) = 4 + a
For a = 4
F(1) = -4 + a = 0
F (-1) = 4 + a = 8
F(0) =- 4
For a > 4 graph will shift upwords

A = - 4
F(1) = -4 + a = -8
F(-1) = 0
F(0) = -4
for -4 < a < 4.
Graph will cut real x – anisdt 3 point

Q48. Let $f: (0,\infty) \to \mathbb{R}$ be given by

$$f(x) = \int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}.$$

Then

(A) $f(x)$ is monotonically increasing on $[1, \infty)$

(B) $f(x)$ is monotonically decreasing on $(0, 1)$

(C) $f(x) + f(1/x) = 0$, for all $x \in (0, \infty)$

(D) $f(2^x)$ is an odd function of x on $\mathbb{R}$

Sol. $f(x) \int_{1/x}^{x} \frac{e^{t+\frac{1}{t}}}{t} dt$

$f'(x) \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} \frac{d(x)}{dx} - \frac{e^{-\left(\frac{1}{x}+x\right)}}{\left(\frac{1}{x}\right)} \left\{\frac{d}{dx}\left(\frac{1}{x}\right)\right\}$

$= e^{-\left(x+\frac{1}{x}\right)} \left[\frac{1}{x} + \frac{1}{x}\right]$

$= \frac{2}{x} e$

Now, $e^{-\left(x+\frac{1}{x}\right)}$ is always positive

and $\frac{2}{x} > 0$ for $x \in [1, \infty]$

Now $f\left(\frac{1}{x}\right) = \int_{x}^{\frac{1}{x}} \frac{e^{t+\frac{1}{t}}}{t} dt = - f(x)$.

and $f(2^x) = \int_{\frac{1}{2^x}}^{2^x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt = - f(2^x)$

$f(2^{-x}) = - f(2^x) \Rightarrow$ odd function

Q49. Let $f: (-\pi/2, \pi/2) \to \mathbb{R}$ be given by

$f(x) = (\log(\sec x + \tan x))^3$

Then

(a) $f(x)$ is an odd function

(b) $f(x)$ is a one-one function

(c) $f(x)$ is an onto function

(d) $f(x)$ is an even function

Sol. $F(x) = (\log(\sec x + \tan x))^3$

Now, $\sec^2 x - \tan^2 x = 1$

$(\sec x - \tan x) = 1/(\sec x + \tan x) = (\sec x + \tan x)^{-1}$

$\therefore f(-x) = (\log(\sec x - \tan x))^3$

$= [-\log(\sec x + \tan x)]^3$

$= -(\log(\sec x + \tan x))^3 = f(x)$

$\therefore f(-x) = -f(x) \Rightarrow$ odd function.
$F'(x) = 3(\log(\sec x - \tan x))^2 \sec^2 x \sec^2 x + \sec x \tan x/(\sec x + \tan x)$

$F'(x) = 3\underbrace{(\log(\sec x - \tan x))^3}_{positive} \underbrace{\sec x}_{positive\ for\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$

$\therefore f'(n) > 0 \Rightarrow f^n$ is increasing $\Rightarrow$ one to one f^n

Ans. A, B, C

Q50. Let $f : [a, b] \to [1, \infty)$ be a continuous function and let $g : \mathbb{R}$ be defined as:

$$g(x) = \begin{cases} 0 & \text{if } x < a, \\ \int_a^x f(t)dt & \text{if } a \le x \le b, \\ \int_a^b f(t)dt & \text{if } x > b. \end{cases}$$

Then
(a) g(x) is continuous but no differentiable at a
(b) g(x) is differentiable on $\mathbb{R}$
(c) g(x) is continuous but not differentiable at b
(d) g(x) is continuous and differentiable at either a or b but not both

Sol. at x = q $\quad g(q^-) = 0$
$\quad g(a^+) = \int_a^a + (t)dt \to$
at (x = b) $\quad g(b^-) = \int_a^b f(t)\,dt$
$\quad g(b^+) = \int_a^b f(t)\,dt$
so, g (at) = g (a⁺) and g (b⁻) = g (b⁺)
so, g (x) is wntinuous at x = 'a' and 'b'

$$g'(x) = \begin{cases} 0\ if\ x < q \\ f(x) if\ a \le x \le b \\ 0\ if\ x > b \end{cases}$$

$\therefore g'(x)$ is not differentiable at x = a and 'b'

SECTION – 2: (One Integer Value Correct Type)

Q51. Let a, b, and c be three non-coplanar unit vectors such that the angle between every pair of them is $\pi/3$. If a x b + b x c = pa + qb + rc, where p, q and r are scalars, then the value of $p^2 + 2q^2 + r^2/q^2$ is

Sol. $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\,\vec{a} + q\,\vec{b} + r\,\vec{c}$
Taking dot product with B
$\vec{b}.(\vec{a} \times \vec{b}) + \vec{b}(\vec{b} \times \vec{c}) = p\,\vec{a}.\vec{b} + q\,\vec{a}.\vec{b}\,q\,\vec{b}.\vec{b} + r\,\vec{c}.\vec{b}$
$0 = \frac{p}{2} + q + \frac{r}{2}$
Taking dot with $\vec{a}$
$\vec{a}.(\vec{a} \times \vec{b}) + \vec{a}.(\vec{b} \times \vec{c}) = p\,\vec{a}.\vec{a} + q\,\vec{a}.\vec{a} + q\,\vec{b}.\vec{a} + r\,\vec{c}.\vec{a}$

$0 + a \cdot (\vec{b} \times \vec{c}) = p + \frac{q}{2} + \frac{r}{2}$ (2)

Similarity with $\vec{c}$

$\vec{c} \cdot (\vec{a} \times \vec{b}) = \frac{p}{2} + \frac{q}{2} + r$ (3)

LHS of (2) & (3) are same

So :

$P + q\frac{1}{2} = \frac{r}{2} = \frac{p}{2} + ½ + r$

$p/2 = r/2$

$p = 2 = r/2$

$p = r$ (4)

from (1) (2) (4)

$p + q = 0$ $\quad\quad p = -q$

$r = -q$

putting value in q :

$= \frac{q^2 + 2q^2 + q^2}{q^2} = 4$

Q52. Let $f : [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation

$F(x) = 10 - x/10$

Is

Q53. For a point P in the plane, let d_1 (P) and d_2 (P) be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \le d^1(P) + d^2(P) \le 4$ is

Q54. The largest value of the non-negative integer a for which

$$\lim_{x \to 1} \left\{\frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$$

Is

Sol. $\lim_{x \to 1} \left\{\frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1}\right\}^{\frac{1-x}{1-\sqrt{x}}}$

$$\lim_{x \to 1} \left\{\frac{-ax + (\sin(x-1) + a + x - 1) - x + 1}{x + \sin(x-1) - 1}\right\}^{\frac{1-x}{1-\sqrt{x}}}$$

$$\lim_{x \to 1} \left\{1 + \frac{a - ax + 1(x-1)}{x + \sin(x-1) - 1}\right\}^{\frac{(1-\sqrt{x})^2}{1-\sqrt{x}}}$$

$$\lim_{x \to 1} \left\{1 + \frac{a(1-x) + 1(1-x)}{(x-1) + \sin(x-1)}\right\}^{\frac{(1+\sqrt{x})(1-\sqrt{x})}{(1-\sqrt{x})}} \left(\begin{matrix} since \\ x \ne 1 \end{matrix}\right)$$

$$\lim_{x \to 1} \left\{1 + \frac{(a+1)(1-x)}{(x-1) + \sin(x-1)}\right\}^{1+\sqrt{x}}$$

$$\lim_{x \to 1} \left\{ 1 + \frac{\frac{3(a+1)(1-x)}{(x-1)}}{\frac{1+\sin x-1}{x-1}} \right\}^{111}$$

$$\left\{ 1 + \frac{-^{1}(a+1)}{\lim_{\lambda \to 1} \frac{\sin \lambda - 1}{\lambda - 1} + 1} \right\}^{2}$$

Now, $\left\{ 1 + \frac{-(a+1)}{1+1} \right\}^2 = \frac{1}{4}$

$\left\{ 1 + \frac{2-a-1}{2} \right\}^2 = \frac{1}{4}$

$\frac{1-a}{2} = \pm \frac{1}{2}(1 - a = \pm 1)a = a$ or $a = 2$

Largest value .2

Q55. Let $n \geq 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is

Sol. Draw figure for cases $n = 3, 4, 5$ At 5 we will get the required condition

Q56. Let a, b, c, be positive integers such that b/a is an integer. If a, b, c, are in geometric progression and the arithmetic mean of a, b, c is b +2, then the value of

$a^2+a-14/a+1$

is

sol. a b c are in G.P

B = ar   r is integer

C = ar^2

Now

A = ar + ar^2 = 3 (ar b + 2)(given)

A + ar + ar^2 = 3 ar + 6

$Ar^2 - 3$ ar + (a - 6)

H is quad ratio in r

R = $\frac{2a \pm \sqrt{4a^2 - 4(a-6)(a)}}{2a}$

R = $1 \pm \frac{2a \pm \sqrt{4a^2 - 4a^2 + 24a}}{2a}$

= $1 \pm \sqrt{\frac{24a}{4a2}} = 1 \pm \sqrt{\frac{6}{a}}$

For r to be integer

$\sqrt{\frac{6}{a}}$should be integer

Which is possible when a = 6

$\frac{a^2+a-14}{a+1} = \frac{42-14}{7} = 4.$

Q57.
Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. Then the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is

Sol. Possible values of n_1 to fulfill above conditions is 1 & 2

For $n_1 = 2$

$n_2 = 3$ $n_3 = 4$ $n_4 = 5$ $n_5 = 6$+

Is the only possibility

For $n_1 = 1$

Possible cases

n_1	n_2	n_3	n_4	n_5
1	2	3	5	9
1	2	3	6	8
1	2	3	4	10
1	2	4	5	8
1	2	4	6	7
1	3	4	5	7

Hence total 7distint arrangements

Q58. Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h: \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} \max \quad \{f(x), g(x)\} & \text{if } x \le 0, \\ \min \quad \{f(x), g(x)\} & \text{if } x > 0. \end{cases}$$

The number of points at which h(x) is not differentiable is

Sol. f(x) = |x| + 1
g(x) = x^2 + 1
So, clearly from graph h(x) is not differentiable at x = -1, 0, 1
⇒ Answer is 3

Q59. The slope of the tangent of the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is

Sol. . $(y - x^5)^2 = x(1 + x^2)^2$

Diff w.r.t x

$2(y - x^5)\left[\frac{dy}{dx} - 5x^4\right] = x[2(1 + x^2)2x] + (1 + x^2)^2$

At (1, 3)

$2(3 - 1)\left[\frac{dy}{dx} - 5\right] = 1[2*2*2(1)] + (1 + 1)^2$ $4\left[\frac{dy}{dx} - 5\right] = 8 + 4 = 12$

$\Rightarrow \frac{dy}{dx} = 8$

Q60. The value of

$\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}(1-x^2)^5\right\} dx$

I8

Sol. $\int_0^1 4x^3 \{\frac{\partial^2}{dx^2}(1-x^2)^5\}dx$

$\frac{\partial^2}{\partial x^2}(1-x^2)^5 = \frac{\partial}{\partial x}\frac{\partial}{\partial x}(1-x^2)^5$

$= \frac{\partial}{\partial x}(5(1-x^2)^4(-2x))$

$= -10[x\,4(1-x^2)^3(-2x) + (1-x^2)^4]$

$= -10(1-x^2)^3[-8x^2 + (1-x^2)]$

$= -10(1-x^2)^3[1-9x^2]$

Now $\int_0^1 4x^3(-10)(1-x^2)^3[1-9x^2]\,dx$

$-40\int_0^1 x^3(1-x^2)^3(1-9x^2)\,dx$

λet $x^2 = t$ $\quad x \to 0\ t \to 0$

$2x\,dx = dt$ $\quad x \to 1\ t \to 1$

$x\,dx = \frac{dt}{2}$

Above becomes

$-20\int_0^1 t(1-t)^3(1-9t)\frac{dt}{2}$

$-20\int_0^1 t\,t(1-t^3-3t+3t^2)(1-9t)\,dt$

$= -20\int_0^1(1-t^3-3t^2+3t^3)(1-9t)\,dt$

$= -20\int_0^1(t-t^4-3t^2+3t^3-9t^2+9t^5+27t^4-27t^3)dt$

$= -20\int_0^1(t-12t^2+30t^3-28t^4+9t^5)dt$

$= -20\left(\frac{t^2}{2}-4t^3+\frac{15t^4}{2}-\frac{28t^5}{5}+\frac{3t^6}{2}\right)$

$= -20[(1/2-4+15/2-28/3+3/2)-(0)]$

$=-190+192$

$=2$