Matrices and Determinants - Solutions

Sol 1.

We should have,

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$$

$\Rightarrow 1\,(-4k + 6) - k\,(-12 + 4) + 3\,(9 - 2k) = 0$

$\Rightarrow -2k + 33 = 0$

$\Rightarrow k = 33/2$

Substituting $k = 33/2$ and putting $x = b$, where $\in Q$, we get the system as

$33y + 6z = -2b$ .... (1)

$33y - 4z = -6b$ .... (2)

$3y - 4z = -2b$ .... (3)

(1) – (2) $\Rightarrow 10z = 4b \Rightarrow z = 2/5\ b$

(1) $\Rightarrow 33y = -2b - 12b/5 = -22b/5 \Rightarrow y = -2b/15$

$\therefore$The solution is $x = b$, $y = -2b/15$, $z = 2b/5$

Sol 2.

The given determinant, on expanding along R_1, we get

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a\,(bc - a^2) - b\,(b^2 - ac) + c\,(ab - c^2)$$

$= 3abc - a^3 - b^3 - c^3 = -\,(a^3 + b^3 + c^3 - 3abc)$

$= -\,(a + b + c)\,[a^2 + b^2 + c^2 - ab - bc - ca]$

$= -1/2\,(a + b + c)\,[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca]$

$= -1/2\,(a + b + c)\,[(a - b)^2 + (b - c)^2 + (c - a)^2]$

As $a, b, c > 0$

$\therefore a + b + c > 0$

Also $a \neq b \neq c$ $\therefore (a - b)^2 + (b - c)^2 + (c - a)^2 > 0$ Hence the given determinant is – ve.

Sol 3.

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = xA + B$$

$$\text{L. H. S.} = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

Operation $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & x+1 & x-2 \\ 0 & x-2 & x+1 \\ 0 & x-2 & x+1 \end{vmatrix} + \begin{vmatrix} x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

$$= 0 + \begin{vmatrix} x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

Operating $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$

$$= \begin{vmatrix} x & x+1 & x-2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} = \begin{vmatrix} x & x & x \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix}$$

$$= x\begin{vmatrix} 1 & 1 & 1 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix}$$

= xA + B = R. H. S. Hence Proved.

Sol 4.

The given system of equations is

$3x - y + 4z = 3$

$x + 2y - 3z = -2$

6x + 5y + λz = -3

Then D = $\begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix}$

= 3 (λ + 15) + 1 (λ + 18) + 4 (5 – 12) = 7 λ + 35 = 7 (λ + 5)

For unique sol., D ≠ 0

∴ λ + 5 ≠ 0 or λ ≠ -5

∴ System has unique SolFor λ ≠ - 5

For λ = - 5, D = 0

Then

$$D_1 = \begin{vmatrix} 3 & -1 & 4 \\ -2 & 2 & -3 \\ -3 & 5 & -5 \end{vmatrix}$$

= 3 (-10 + 15) + 1 (10 – 9) + 4 (-10 + 6)

= 15 + 1 – 16 = 0

$$D_2 = \begin{vmatrix} 3 & 3 & 4 \\ 1 & -2 & -3 \\ 6 & -3 & -5 \end{vmatrix}$$

= 3 (10 – 9) – 3 (-5 + 18) + 4 (-3 + 12) = 3 – 39 + 36 = 0

$$D_3 = \begin{vmatrix} 3 & -1 & 3 \\ 1 & 2 & -2 \\ 6 & -5 & -3 \end{vmatrix}$$

= 3 (-6 + 10) + 1 (-3 + 12) + 3 (5 – 12) = 12 + 9 – 21 = 0

$D_1 = D_2 = D_3 = 0$

∴ Infinite many solutions.

Let z = k then equations becomes

3x – y = 3 – 4k

x + 2y = 3k – 2

on solving we get

x = 4 – 5k/7, y = 13k – 9/7, z = k

This satisfies the third equation.

Sol 5.

On L. H. S. = D, applying operations $C_2 \to C_2 + C_1$ and $C_3 \to C_3 + C_2$ and using ${}^{n}C_r + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$, we get

$$D = \begin{vmatrix} {}^{x}C_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^{y}C_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^{=}C_r & {}^{=+1}C_{r+1} & {}^{=+1}C_{r+2} \end{vmatrix}$$

Operating $C_3 + C_2$ and using the same result, we get

$$D = \begin{vmatrix} {}^{x}C_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^{y}C_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^{=}C_r & {}^{=+1}C_{r+1} & {}^{=+2}C_{r+2} \end{vmatrix} = \text{R H S}$$

Hence Proved

Sol 6.

The system will have a non – trivial solution if

$$\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

Expanding along C_1, we get

$\Rightarrow (28 - 21) \sin 3\theta - (-7 - 7) \cos 2\theta + 2(-3 - 4) = 0$

$\Rightarrow 7 \sin 3\theta + 14 \cos 2\theta - 14 = 0$

$\Rightarrow \sin 3\theta + 2 \cos 2\theta - 2 = 0$

$\Rightarrow 3 \sin - 4 \sin^3 \theta + 2 (1 - 2 \sin^2 \theta - 2 = 0$

$\Rightarrow 4 \sin^3 \theta + 4 \sin^2 \theta - 3 \sin \theta = 0$

$\Rightarrow \sin \theta (2 \sin \theta - 1) (2 \sin \theta + 3) = 0$

Sin $\theta = 0$ or $\sin \theta = 1/2$

($\sin \theta = -3/2$ not possible)

$\Rightarrow \theta = n\pi$ or $\theta = n\pi + (-1)^n \pi/6, n \in Z$.

Sol 7.

We have

$$\Delta a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^2 & 3n^2-3n \end{vmatrix}$$

NOTE THIS STEP

Then $\sum_{a=1}^{n} \triangle a = \begin{vmatrix} (1-1) & n & 6 \\ (1-1)^2 & 2n^2 & 4n-2 \\ (1-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix} + \begin{vmatrix} (2-1) & n & 6 \\ (2-1)^2 & 2n^2 & 4n-2 \\ (2-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$+ \begin{vmatrix} (n-1) & n & 6 \\ (n-1)^2 & 2n^2 & 4n-2 \\ (n-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} 1+2+3+\ldots\ldots+(n-1) & n & 6 \\ 1^2+2^2+3^2+\ldots\ldots+(n-1)^2 & 2n^2 & 4n-2 \\ 1^3+2^3+3^3+\ldots\ldots+(n-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(n-1)}{2} & n & 6 \\ \frac{n(n-1)(2n-1}{6} & 2n^2 & 4n-2 \\ \left(\frac{n(n-1)}{2}\right)^2 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{n^2(n-1)}{12} \begin{vmatrix} 6 & 1 & 6 \\ 2(2n-1) & 2n & 2(2n-1) \\ 3n(n-1) & 3n^2 & 3n(n-1) \end{vmatrix}$$

(Taking n(n – 1)/12 common from C_1 and n from C_2)

= 0 (as C_1 and C_3 are identical)

Thus, $\sum_{a=1}^{n} \triangle a = 0 \Rightarrow \sum_{a=1}^{n} \triangle a$ = c (a constant)

Where c = 0

Sol 8.

Given that A, B, C are integers between 0 and 9 and the three digit numbers A 28, 3 B9 and 62C are divisible by a fixed integer k.

Now, D = $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$

On operating $R_2 \to R_2 + 10\, R_3 + 100\, R_1$, we get

$$= \begin{vmatrix} A & 3 & 6 \\ A28 & 3B9 & 62C \\ 2 & B & 2 \end{vmatrix} = \begin{vmatrix} A & 3 & 6 \\ kn_1 & kn_2 & kn_3 \\ 2 & B & 2 \end{vmatrix}$$

[As per question A28, 3 B9 and 62 C are divisible by k, therefore,

A 28 = kn_1

3 B9 = kn_2

62 C = kn_3]

$$= k\begin{vmatrix} A & 3 & 6 \\ n_1 & n_2 & n_3 \\ 2 & B & 2 \end{vmatrix}$$ = k x some integral value.

⇒ D is divisible by k.

Sol 9.

Consider $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$

Operating $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ we get

$$\begin{vmatrix} p-a & -(q-b) & c \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

Taking (p – q), (q – b) and (r – c) common from (C_1, C_2 and C_3 resp. we get

$$\Rightarrow (p - a)(q - b)(r - c)\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \dfrac{a}{p-a} & \dfrac{b}{q-b} & \dfrac{r}{r-c} \end{vmatrix} = 0$$

Expanding along R_1

⇒ (p – a) (q – b) (r – c) [1 (r/r – c + b/q – b) + a/p – a]=0

As given that p ≠ a, q ≠ b, r ≠ c

∴ r/r – c + b/q – b + a/p – a = 0

⇒ r/r – c + q – (q – b)/q – b + p – (p – a)/p – a = 0

⇒ r/r – c + q/ q – b – 1 + p/p – a – 1 = 0

⇒ p/p – a + q/q – b + r/r – c = 2

Sol 10.

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$

$$= n!\,(n+1)!\,(n+2)! \begin{vmatrix} 1 & n+1 & (n+2)(n+1) \\ 1 & n+2 & (n+3)(n+2) \\ 1 & n+3 & (n+4)(n+3) \end{vmatrix}$$

Operating $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we get

$$D = (n!)^3 (n+1)^2 (n+2) \begin{vmatrix} 1 & n+1 & n^2+3n+2 \\ 0 & 1 & 2n+4 \\ 0 & 1 & 2n+6 \end{vmatrix}$$

Operating $R_3 \to R_3 - R_2$

$$D = (n!)^3 (n+1)^2 (n+2) \begin{vmatrix} 1 & n+1 & n^2+3n+2 \\ 0 & 1 & 2n+4 \\ 0 & 0 & 2 \end{vmatrix}$$

$= (n!)^3 (n + a)^2 (n + 2)\ 1\ [2]$

$\Rightarrow D/(n!)^3 = 2 (n + 1)^2 (n + 2)$

$\Rightarrow D/(n!)^3 - 4 = 2(n^3 + 4n^2 + 5n + 2) - 4$

$= 2\,(n^3 + 4n^2 + 5n) = 2n\,(n^2 + 4n + 5)$

$\Rightarrow D/(n!)^3 - 4$ is divisible by n.

Sol 11.

Given that $\lambda, \alpha \in R$ and system of linear equations

$\lambda x + (\sin\alpha)\, y + (\cos\alpha)\, z = 0$

$x + (\cos\alpha)\, y + (\sin\alpha)\, z = 0$

$-x\, (\sin\alpha)\, y - (\cos\alpha)\, z = 0$

Has a non trivial solution, therefore D = 0

$$\Rightarrow \begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

$\Rightarrow \lambda\,(-\cos^2\alpha - \sin^2\alpha) - \sin\alpha\,(-\cos\alpha + \sin\alpha) + \cos\alpha\,(\sin\alpha + \cos\alpha) = 0$

$\Rightarrow -\lambda + \sin\alpha\cos\alpha - \sin^2\alpha + \sin\alpha\cos\alpha + \cos^2\alpha = 0$

$\Rightarrow \lambda = \cos^2\alpha - \sin^2\alpha + \sin\alpha\cos\alpha$

$\Rightarrow \lambda = \cos 2\alpha + \sin 2\alpha$

For $\lambda = 1$, $\text{Cos } 2\alpha + \sin 2\alpha = 1$

$1/\sqrt{2}\cos 2\alpha + 1/\sqrt{2}\sin 2\alpha = 1/\sqrt{2}$

$\Rightarrow \cos 2\alpha\cos\pi/4 + \sin 2\alpha\sin\pi/4 = 1/\sqrt{2}$

$\Rightarrow \text{cps}\,(2\alpha - \pi/4) = \cos\pi/4$

$\Rightarrow 2\alpha - \pi/4 = 2n\pi \pm \pi/4$

$\Rightarrow 2\alpha = 2\pi r + \pi/4 + \pi/4;\ 2n\pi - \pi/4 + \pi/4$

$\Rightarrow \alpha = n\pi + \pi/4 \text{ or } n\pi$

Sol 12.

$$\text{L. H. S} = \begin{vmatrix} \cos A\cos P + \sin A\sin P & \cos(A-Q) & \cos(A-R) \\ \cos B\cos P + \sin B\sin P & \cos(B-Q) & \cos(B-R) \\ \cos C\cos P + \sin C\sin P & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \cos P = \begin{vmatrix} \cos A & \cos(A-Q) & \cos(A-R) \\ \cos B & \cos(B-Q) & \cos(B-R) \\ \cos C & \cos(C-Q) & \cos(C-R) \end{vmatrix} + \sin P \begin{vmatrix} \sin A & \cos(A-Q) & \cos(A-R) \\ \sin B & \cos(B-Q) & \cos(B-R) \\ \sin C & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

Operating; $C_2 \to C_2 - C_1 (\cos Q)$; $C_3 \to C_3 - C_1 (\cos R)$ on first determinant and $C_2 \to C_3 - (\sin Q) C_1$ and $C_3 \to C_3 - (\sin R) C_1$ on second determinant, we get

$$= \cos P \begin{vmatrix} \cos A & \sin A \sin Q & \sin A \sin R \\ \cos B & \sin B \sin Q & \sin B \sin R \\ \cos C & \sin C \sin Q & \sin C \sin R \end{vmatrix} = \cos P \begin{vmatrix} \sin A & \cos A \cos Q & \cos A \cos R \\ \sin B & \cos B \cos Q & \cos B \cos R \\ \sin C & \cos C \cos Q & \cos C \cos R \end{vmatrix}$$

$$= \cos P \sin Q \sin R \begin{vmatrix} \cos A & \sin A & \sin A \\ \cos B & \sin B & \sin B \\ \cos C & \sin C & \sin C \end{vmatrix}$$

$$+ \sin P \cos Q \cos R \begin{vmatrix} \sin A & \cos A & \cos A \\ \sin B & \cos B & \cos B \\ \sin C & \cos C & \cos C \end{vmatrix}$$

$= 0 + 0$ [Both determinants become zero as $C_2 \equiv C_3$] $= 0 =$ R. H. S.

<u>Sol 13.</u>

Let us denote the given determinant by Δ. Taking

1/a (a + d) (a + 2d) as common from

R_1, 1/(a + d) (a + 2d) (a + 3d) from R_2 and

1/(a + 2d) (a + 3d) (a + 4d) from R_3 we get

$\Delta = 1/a\,(a + d)^2 (a + 2d)^3 (a + 3d)^2 (a + 4d)\, \Delta_1$

Where

$$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+4d & a+2d \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(2d) & d & d \\ (a+3d)(2d) & d & d \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get

$$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(2d) & d & d \\ 2d^2 & 0 & 0 \end{vmatrix}$$

Expanding along R_3, we get

$$\Delta_1 = (2d)^2 \begin{vmatrix} a+2d & a \\ d & d \end{vmatrix} = (2d)^2 (d) (a + 2d - a) = 4d^4$$

Thus, $\Delta = 4d^4/a\ (a + d)^2 (a + 2d)^3 (a + 3d)^2 (a + 4d)$.

Sol14.

Given that a, b, c are pth, qth and rth terms of an H. P.

⇒ 1/a, 1/b, 1/c are pth, qth, rth terms of an A. P.

$$\therefore \left.\begin{aligned} \frac{1}{a} &= A + (p-1)D \\ \frac{1}{b} &= A + (q-1)D \\ \frac{1}{c} &= A + (r-1)D \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now given determinant is

$$\Delta = \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abc \begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Substituting the values of 1/a, 1/b, 1/c from (1), we get

$$= abc \begin{vmatrix} A + (p-1)D & A + (q-1)D & A + (r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$ Operating $R_1 \to R_1 - (A - D) R_3 - D R_2$, we get

$$\Delta = abc \begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Sol 15.

$R_2 \rightarrow R_2 + R_3$,

$$= 0 \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ 2\sin\theta\cos\frac{2\pi}{2} & 2\cos\theta\cos\frac{2\pi}{3} & 2\sin 2\theta\cos\frac{4\pi}{3} \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0$$

Sol 16.

Given that A_T A = I

Where A = $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ ∴ A_T = $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

∴ A^T A = I

$$\Rightarrow \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ca & ab+bc+ca & a^2+b^2+c^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow a^2 + b^2 + c^2 = 1$ . . . . (1)

And ab + bc + ca = 0 . . . . (2)

abc = 1 (given) . . . . (3)

Now $a^3 + b^3 + c^3 - 3abc$

$= (a + b + c) (a^2 + b^2 + c^2 - ab - bc - ca)$

$\Rightarrow a^3 + b^3 + c^3 = (a + b + c) [1 - 0] + 3 \times 1$ [Using (1), (2) and (3)]

$\Rightarrow a^3 + b^3 + c^3 = a + b + c + 3$

Now $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow (a + b + c +)^2 = 1$

$\Rightarrow (a + b + c) = 1$

[$\because$ $(a + b + c) \neq -1$ as a, b, c all are + ve numbers]

$\therefore$we get

$a^3 + b^3 + c^3 = 4$

ALTERNATE SOLUTION

Given that $A^T A = I$

$\Rightarrow |A^T A| = |A^T|\,|A| = |A|\,|A| = 1$ [$\because |I| = 1$]

$\Rightarrow |A|^2 = 1$ .... (1)

From given matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

$|A| = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = a^3 + b^3 + c^3 - 3abc$ .... (2)

$\therefore (a^3 + b^3 + c^3 - 3abc = 1$ or -1

But for a^3, b^3 c^3 using AM $\geq$ GM

We get $a^3 + b^3 + c^3/3 \geq \sqrt[3]{a^3 b^3\ c^3}$

$\therefore$We must have

$a^3 + b^3 + c^3 - 3abc = 1$

$\Rightarrow a^3 + b^3 + c^3 = 1 + 3 \times 1 = 4$ [Using $abc = 1$]

Sol 17.

We are given that $MM^T = I$ where M is a square matrix of order 3 and det M = 1.

Consider $\det(M - I) = \det(M - MM^T)$ [Given $MM^T = I$]

$= \det[M(I - M^T)]$ [$\because |AB| = |A|\,|B|$]

$= -(\det M)(\det(M^T - I))$

$= -1[\det(M^T - I)]$ [$\because \det(M) = 1$]

$= -\det(M - I)$

$[\because \det(M^T - I) = \det[(M - I)^T] = \det(M - I)]$

$\Rightarrow 2\det(M - I) = 0 \Rightarrow \det(M - I) = 0$ Hence Proved

Sol18.

Given that, A = $\begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, U = $\begin{bmatrix} f \\ g \\ h \end{bmatrix}$

And AX = U has infinite many solution.

$\Rightarrow |A| = 0 = |A_1| = |A_2| = |A_3|$

Now, $|A| = 0$

$\Rightarrow \begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix} = a(bc - bd) - 1(c - d) = 0$

$\Rightarrow (ab - 1)(c - d) = 0$

$\Rightarrow ab = 1$ or $c = d$ ………………………………… (1)

And $|A_1| = \begin{vmatrix} f & 1 & 0 \\ g & b & d \\ h & b & c \end{vmatrix} = 0$

$\Rightarrow f(bc - bd) - 1(gc - hd) = 0$

$\Rightarrow f\,b\,(c - d) = gc - hd$ …………………………………… (2)

$|A_2| = \begin{vmatrix} a & 1 & f \\ 1 & b & g \\ 1 & b & h \end{vmatrix} = 0$

$\Rightarrow a(gc - hd) - f(c - d) = 0 \Rightarrow a(gc - hd) = f(c - d)$

$|A_3| = \begin{vmatrix} a & 1 & f \\ 1 & b & g \\ 1 & b & h \end{vmatrix} = 0$

$\Rightarrow a(bh - bg) - 1(h - g) + f(b - d) = 0$

$\Rightarrow ab(h - g) - (h - g) = 0$

$\Rightarrow ab = 1$ or $h = g$ ………………………………… (3)

Taking $c = d \Rightarrow h = g$ and $ab \neq 1$ (from (1), (2) and (3))

Now taking BX = V

Where B = $\begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix}$, V = $\begin{vmatrix} a^2 \\ 0 \\ 0 \end{vmatrix}$

Then |B| = $\begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix} = 0$

[∵ In vew pf c = d and g = h, C_2 and C_2 and C_3 are indentical

⇒ BX = V has no unique solution

And $|B_1| = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix} = 0$ (∵ c = d, g = h)

$|B_2| = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix}$ a^2 cf = a^2 d f (∵ c = d)

$|B_3| = \begin{vmatrix} a & 1 & a^2 \\ 0 & d & 0 \\ f & g & 0 \end{vmatrix}$ a^2 d f

⇒ If adf ≠ 0 then $|B_2| = |B_3| \neq 0$

Hence no solution exist.