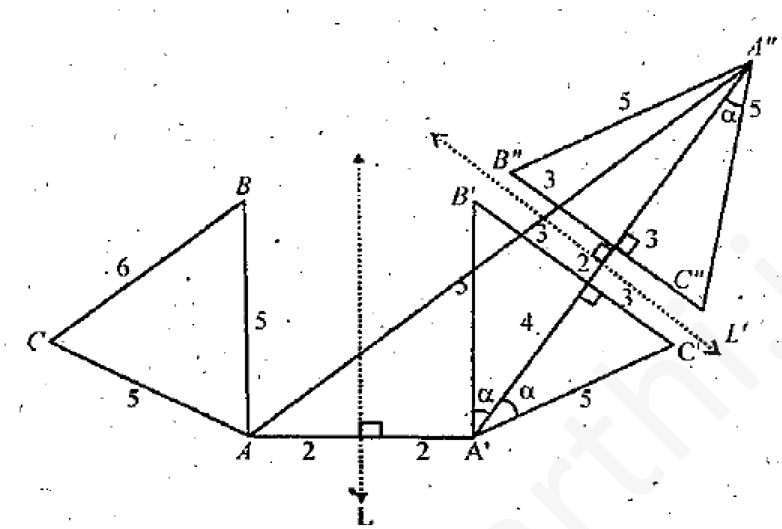
Properties of Triangle – Solutions

Sol 1.

Let L be the line parallel to side AB of Δ ABC, at a distance of 2 cm from AB, in which the first reflection ΔA' B' C' is obtained. Let L' be the second line parallel to B' C', at a distance of 2 cm from B' C', in which reflection of ΔA' B' C' is taken as ΔA" B" C".

In figure, size of ΔA" B" C" is same to the size of ΔA' B' C'.



From figure AA' = 4cm and A'A" = 12 cm. So to find AA" it suffices to know ∠AA'A", clearly

∠AA'A" = 90° + α where sin α = 3/5

⇒ cos ∠AA'A" = cos (90° + α) = sin α = -3/5 and hence

AA" = √ (AA') ² + (A' A") ² – 2AA' x A' A". cos (90° + α)

= √16 + 144 + 96 x 3/5

= √1088/5 = 8√17/5 cm.

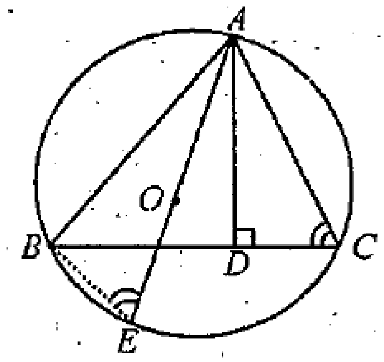
Sol 2.

(a) In radius of the circle is given by

r = (s – b) tan B/2 = (a + b + c/2 – b) tan π/4 = a + c – b/2

2 r = a + c – b ⇒ Diameter = BC + AB – AC

(b) Given a ΔABC in which AD ⊥ BC, AE is diameter of circumcircle of Δ ABC.



To prove:

AB x AC = AE x AD

Construction: Join BE

Proof: ∠ABE = 90°   (∠ in a semi-circle)

Now in Δ's ABE and ADC

∠ABE = ∠ADC   (each 90°)

∠ AEB = ∠ACD   (∠'s in the same segment)

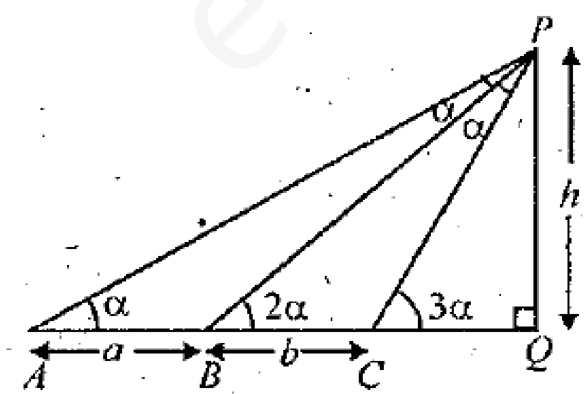
∴ ΔABE ~ ΔADC   (by AA similarity)

⇒ AB/AD = AE/AC

⇒ AB x AC x = AD x AR   (Proved)

Sol 3.

(a) By exterior angle theorem, in the adjacent fig.



∠APB = ∠BPC = α

Also in Δ ABP, ∠BAP = ∠APB = α

⇒ AB = PB = a

Applying sine law in ΔPBC, we get

$\Rightarrow$ a /sin (180° - 3α) = b/sin α = PC/sin 2 α .... (1)

$\Rightarrow$ a/3 sin α – 4 $\sin^3$ α = b/sin α = PC/2 sin α cos α

$\Rightarrow$ a/3 – 4 $\sin^2$ α = b/1 = PC/2 cos α

$\Rightarrow$ 3 – 4 $\sin^2$ α = a/b $\Rightarrow$ $\sin^2$ α = 3b – a/4b

$\Rightarrow$ cos ^{2} α = b + a/4b

$\Rightarrow$ cos α = 1/2 √b + a/b

Also PC = 2b cos α = √b (a + b)

Now in ΔPCQ

Sin 3 α = h/PC $\Rightarrow$ h = PC (a sin α/b) [Using eqn. (1)]

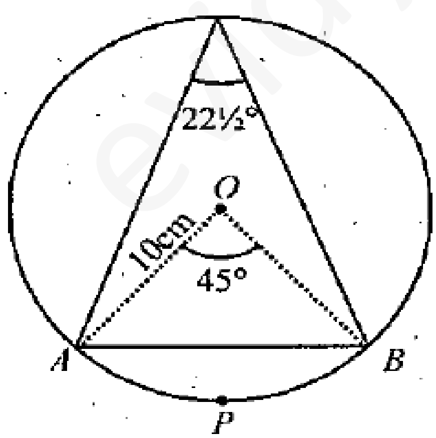
$\Rightarrow$ h = √b (a + b) a/b $\sqrt{\frac{3b-a}{4b}}$

$\Rightarrow$ h = a/2b √ (a + b) (3b – a)

(b) ∵ ∠ACB = $22\frac{1}{2}^\circ$

∴ ∠AOB = 45°

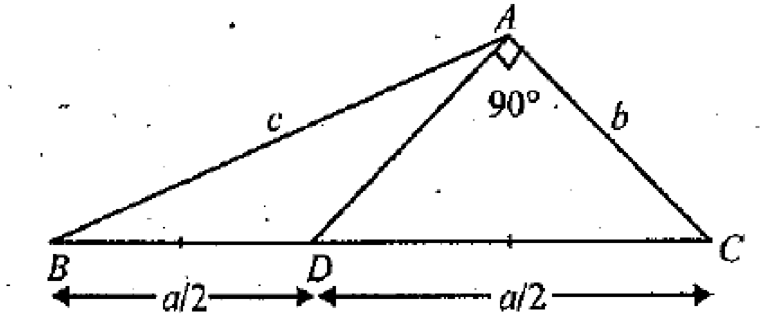
r = 10 cm



Area of the segment APB = Area of the sector AOB – area of Δ AOB

= 1/8 πr^2 – 1/2 x 10 x 10 sin 45° (Using Δ = 1/2 bc sin A) = 3.14 x 100/8 – 50/√2 = 3.91 sq. cm.

Sol 4.



In Δ ACD, cos C = b/a/2 = 2b/a .... (1)

In Δ ABC, cos C = $a^2 + b^2 + c^2/2ab$ .... (2)

From (1) and (2),

$2b/a = a^2 + b^2 - c^2/2ab \Rightarrow b^2 = 1/3\ (a^2 - c^2)$ .... (3)

Also cos A = $b^2 + c^2 - a^2/2bc$

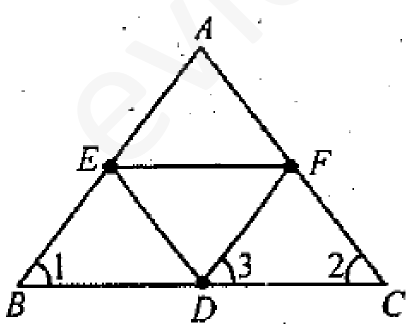
∴ cos A cos C = $b^2 + c^2 - a^2/2bc$ x $2b/a = b^2 + c^2 - a^2/ac$

$= \frac{\frac{1}{3}(a^2 - c^2) + (c^2 - a^2)}{ac} = 2(c^2 - a^2)/3ac$

Sol 5.

Given that AB = AC

∴ ∠1 = ∠2 .... (1)



But AB || DF (given) and BC is transversal

∴ ∠1 = ∠3 .... (2)

From equation (1) and (2)

∠2 = ∠3

⇒ DF = CF .... (3)

Similarly we can prove

DE = BE

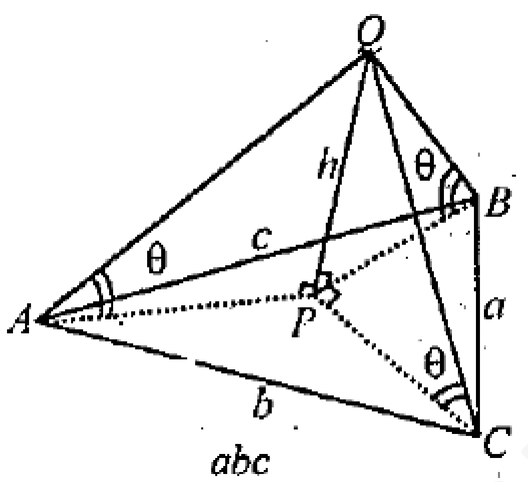
Now, DF + FA + AE +ED = CF + FA + AE + BE

= AC + AB [using equation (3) and (4)]

Sol 6.

(i) Let h be the height of tower PQ.

In Δ APQ tan θ = h/AP ⇒ AP = h/tan θ



Similarly in Δ's BPQ and CPQ we obtain

BP = h/tan θ = CP

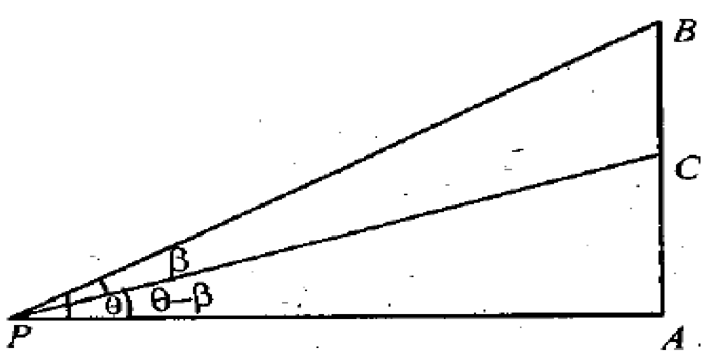
∴ AP = BP = CP

⇒ P is the circum-centrre of Δ ABC with circum radius R = AP = abc/4Δ

∴ h = AP tan θ = abc tan θ/4 Δ

(ii) Given AP = AB x n

⇒ AB/AP = 1/n = tan θ ∴ tan θ = 1/n



Also tan (θ – β) = AC/AP = 1/2 AB/AP = 1/2n

⇒ Tan θ – tan β/ 1+ tan θ tan β = 1/2n ⇒ $\frac{\frac{1}{n} - \tan\beta}{1+\frac{1}{n}\tan\beta} = 1/2n$

⇒ 2n – $2n^2$ tan β = n + tan β

⇒ ($2n^2$ + 1) tan β = n ⇒ tan β = n/$2n^2$ + 1

Sol 7.

As the angles A, B, C of Δ ABC is in AP

∴ Let A = x – d, B = x, C = x + d

But A + B + C = 180° (∠ Sum prop. of Δ)

∴ x – d + x + x + d = 180°

⇒ 3x = 180° ⇒ x = 60° ∴ ∠B = 60°

Now by sine law in Δ ABC, we have

b/ sin B = c/sin C ⇒ sin B/sin C

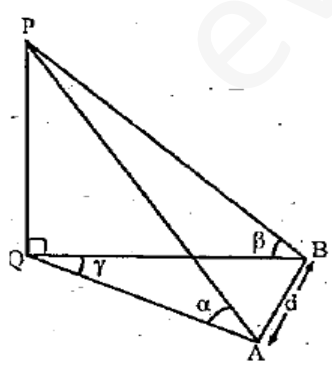
⇒ √3/√2 = sin 60°/sin C [using b: c = √3: √2 and ∠B = 60°]

⇒ √3/√2 = √3/2 sin C ⇒ sin C = 1/√2 = sin 45°

∴ ∠C = 45° ⇒ ∠A = 180° - (∠B + ∠C) = 180° - (60° + 45°) = 75°

Sol 8.

Let ht. of pole PQ be h.



In Δ APQ, tan α = h/AQ

⇒ AQ = h/tan α .... (1)

In Δ BPQ, tan β = h/BQ ⇒ BQ = 4/tan β .... (2)

In Δ ABQ, cos γ = $AQ^2 + BQ^2 – AB^2/2$ AQ BQ

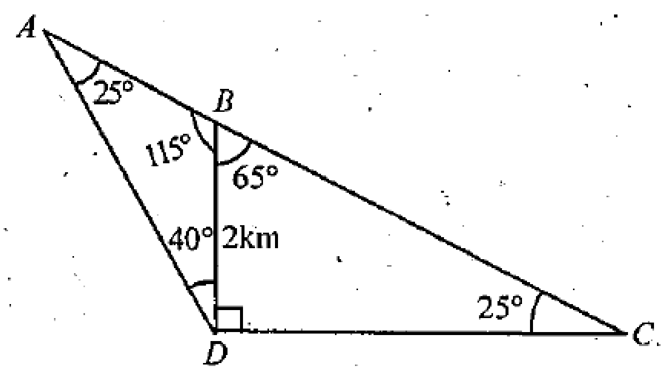
$\therefore \cos\gamma = h^2 \cot^2\alpha + h^2 \cot^2\beta - d^2/2 h^2 \cot\alpha \cot\beta$

$\therefore - 2h^2 \cot\alpha \cot\beta \cos\gamma + h^2 \cot^2\alpha + h^2 \cot^2\beta = d^2$

$\Rightarrow \quad h = d/\sqrt{\cot^2\alpha + \cot^2\beta - 2 \cot\alpha \cot\beta \cos\gamma}$

Sol 9.

According to question figure is as follows



Here, $\angle BDC = 90°$, BD = 2 km

$\angle BDA = 40° \Rightarrow \angle ADC = 130°$

$\therefore \angle DAC = 180° - (25° + 130°) = 25°$

From the figure, in Δ ABD, using sine law

$AD/\sin 115° = BD/\sin 25° \Rightarrow$

$AD = 2 \sin (90° + 25°)/\sin 25° = 2 \cos 25°/\sin 25°$

$\Rightarrow AD = 2 \cot 25° = 2 \sqrt{1/\sin^2 25° - 1} = 2\sqrt{1/ (0.423)^2 - 1} = 4.28$ km

Sol 10.

KEY CONCEPT:

Ex – radii of a ΔABC are $r_1 = \Delta/s - a$, $r_2 = \Delta/s - b$

$r_3 = \Delta/s - c$ As r_1, r_2, r_3 are in H. P. $\therefore 1/r_1, 1/r_2, 1/r_3$ are in AP

$\Rightarrow s - a/\Delta, s - b/\Delta, s - c/\Delta$ are in AP

$\Rightarrow s - a, s - b, s - c$ are in AP

$\Rightarrow -a, -b, -c$ are in AP.

$\Rightarrow a, b, c$ are in A. P.

Sol 11.

Given that in Δ ABC

Cos A + cos B + cos C = 3/2

$\Rightarrow b^2 + c^2 - a^2/2bc \; + \; a^2 + c^2 - b^2/2ac + \; a^2 + b^2 - c^2/2ab = 3/2$

$ab^2 + ac^2 - a^3 + a^2b + bc^2 - b^3 + ac^2 + b^2 c - c^3 = 3abc$

$\Rightarrow ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2 - 6abc = a^3 + b^3 + c^3 - 3abc$

$\Rightarrow a(b - c)^2 + b(c - a)^2 + c(a - b)^2 = (a + b + c/2) [(a - b)^2 + (b - c)^2 + (c - a)^2]$

$\Rightarrow (a + b - c)(a - b)^2 + (b + c - a)(b - c)^2 + (c + a - b)(c - a)^2 = 0$ .... (1)

$a + b > c$

$b + c > a$ {sum of any two sides of a Δ is greater than the third side

As we know that $c + a > b$

∴ Each part on the LHS of eq. (1) has +ve coeff. Multiplied by perfect square, each must be separately zero

∴ $a - b = 0; b - c = 0; c - a = 0 \Rightarrow a = b = c$

Hence Δ is an equilateral Δ

ALTERNATE SOLUTION:

Given that cos A + cos B + cos C = 3/2 in ΔABC

$\Rightarrow 2 \cos A + B/2 \cos A - B/2 = 3/2 - \cos C$

$\Rightarrow 2 \sin C/2 \cos A - B/2 = 3 - 2 \cos C/2$

$\Rightarrow 2 \sin C/2 \cos A - B/2 = 3 - 2(1 - 2 \sin^2 C/2/4 \sin C/2$

$\Rightarrow \cos (A - B/2) = 1 + 4 \sin^2 C/2 \; / \; 4 \sin C/2$

$\Rightarrow \cos (A - B/2) = 1 + 4 \sin^2 C/2 - 4 \sin C/2 + 4 \sin C/2 \; / \; 4 \sin C/2$

$\Rightarrow \cos(A - B/2) = (1 - 2 \sin C/2)^2/4 \sin C/2 + 1$

Which is possible only when

$1 - 2 \sin C/2 = 0 \Rightarrow \sin C/2 = 1/2$

$\Rightarrow C/2 = 30° \Rightarrow C = 60°$

Also then $\cos A - B/2 = 1 \Rightarrow A - B/2 = 0 \Rightarrow A - B=0$ .... (1)

And $A + B = 180° - 60° = 120° \; A + B = 120°$ .... (2)

From (1) and (2) $A = B = 60°$

Thus we get $A = B = C = 60°$ ∴ Δ ABC is an equilateral Δ.

Sol 12.

Given that, in ΔABC,

$b + c/11 = c + a/12 = a + b/13$

Where a, b, c are the lengths of sides, BC, CA and AB respectively.

Let $b + c/11 = c + a/12 = a + b/13 = k$

$\Rightarrow b + c = 11\ k$ .... (1)

$\Rightarrow c + a = 11\ k$ .... (2)

$\Rightarrow a + b = 13\ k$ .... (3)

Adding the above three eqs. We get

$2\ (a + b + c) = 36\ k$

$\Rightarrow a + b + c = 18\ k$ .... (4)

Solving each of (1), (2) and (3) with (4), we get

Now, $\cos A = b^2 + c^2 - a^2/2bc$

$= 36\ k^2 + 25\ k^2 - 49\ k^2/2 \times 6k \times 5k = 12\ k^2/60\ k^2 = 1/5$

$\cos B = c^2 + a^2 - b^2/2ca = 25\ k^2 + 49 - 36\ k^2/2 \times 5\ k \times 7\ k$

$= 38\ k^2/70\ k^2 = 19/35$

$\cos C = a^2 + b^2 - c^2/2ab = 49\ k^2 + 36k^2 - 25k^2/2 \times 7\ k \times 6\ k$

$= 60\ k^2/84\ k^2 = 5/7$

$\therefore \cos A / 1/5 = \cos B / 19/35 = \cos C / 5/7$

$\Rightarrow \cos A / 7/35 = \cos B / 19/35 = \cos C / 25/35$

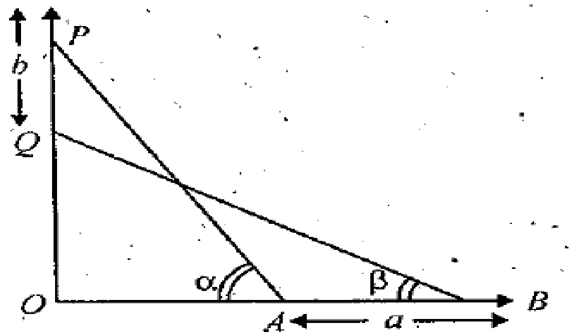
$\Rightarrow \cos A/7 = \cos B/19 = \cos C/25$

Sol 13.

Let the length of the ladder, then

In Δ OQB, $\cos \beta = OB/BQ$

$\Rightarrow OB = \ell \cos \beta$ .... (1)



Similarly in Δ OPA, cos α = OA/PA

⇒ OA = ℓ cos α .... (2)

Now a = OB – OA = ℓ (cos β – cos α) .... (3)

Also from Δ OAP, OP = ℓ sin α

And in OQB; OQ = ℓ sin β

∴ b = OP – OQ = ℓ (sin α – sin β) .... (4)

Dividing eq. (3) by (4) we get

a/b = cos β – cos α/sin α – sin β

$$\Rightarrow \frac{a}{b} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right) - \sin\left(\frac{\alpha-\beta}{2}\right)}$$

⇒ a/b = tan (α + β/2)

Thus , a = b tan (α + β/2) is proved

<u>Sol 14.</u>

Let AD be the median in ΔABC.

Let ∠B = θ then ∠C = 105° - θ

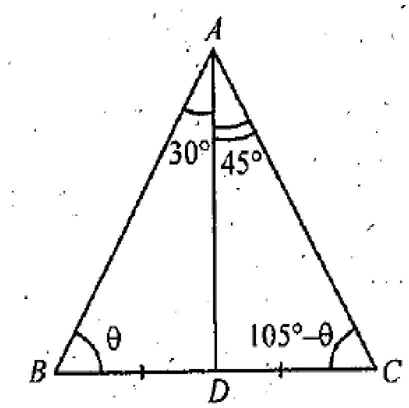
In Δ ABD, using sine law, we get

BD/sin 30° = AD/sin 3 θ ⇒ BD = AD/2 sin θ

In Δ ACD, using sine law, we get

DC/sin 45° = AD/sin (105° - θ) ⇒ DC = AD/√2 sin(105° - θ)

As BD = DC



⇒ AD/2 sin θ = AD/√2 sin (105° - θ)

⇒ sin (90° + 15° - θ) = √2 sin θ

⇒ cos 15° cos θ + sin 15° sin θ = √2 sin θ

⇒ cot θ = √2 – sin 15°/cos 15° = 5 - √3/√3 + 1 = 3√3 – 4

⇒ cosec θ = √1 + cot² θ = √1 + 27 + 16 - 24√3

⇒ cosec θ = 2√11 - 6√3

∴BD = AD/2 sin θ = 1/√11 – 6 √3 x 2√11 - 6√3/2 = 1

∴ BC = 2 BD = 2 units

Sol 15.

We are given that in Δ ABC cos A cos B + sin A sin B sin C = 1

⇒ sin A sin B sin C = 1 – cos A cos B

⇒ sin C = 1 – cos A cos B/sin A sin B

⇒ 1 – cos A cos B/sin A sin B ≤ 1 [∵ sin C ≤ 1]

⇒ 1 – cos A cos B ≤ sin A sin B

⇒ 1 ≤ cos A cos B + sin A sin B

⇒ 1 ≤ cos(A – B)

⇒ 1 ≤ cos(A – B)

But we know cos (A – B) ≤1

∴ We must have cos (A – B) = 1

⇒ A – B = 0

⇒ A = B

∴ $\cos A \cos A + \sin A \sin A \sin C = 1$ [For A = B]

⇒ $\cos^2 A + \sin^2 A \sin C = 1$

⇒ $\sin^2 A \sin C = 1 - \cos^2 A$

⇒ $\sin^2 A \sin C = \sin^2 A$

⇒ $\sin^2 A (\sin C - 1) = 0$

⇒ $\sin A = 0$ or $\sin C = 1$

The only possibility is $\sin C = 1 \Rightarrow C = \pi/2$

∴ $A + B = \pi/2$

But $A = B \Rightarrow A = B = \pi/4$

∴ By Sine law in Δ ABC,

$a/\sin A = b/\sin B = c/\sin C$

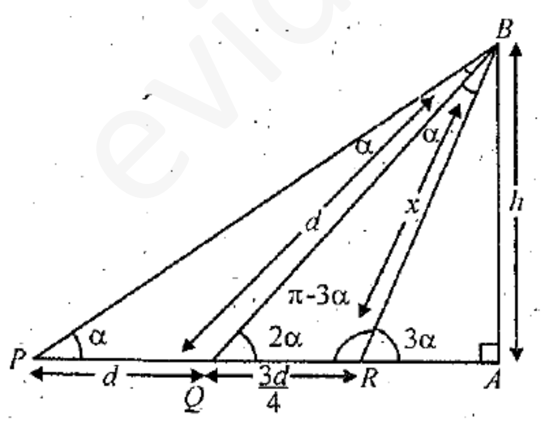
⇒ $a/\sin 45° = b/\sin 45° = c/\sin 90°$

⇒ $a/1/\sqrt{2} = b/1/\sqrt{2} = c/1$

⇒ $a/1 = b/1 = 1/\sqrt{2} \Rightarrow a : b : c = 1 : 1 : \sqrt{2}$

Hence proved the result.

Sol 16.



Let RB = x

∠BQR is ext ∠ of Δ PBQ

∴ $\angle PBQ = 2\alpha - \alpha = \alpha$

Now in Δ PBQ, ∠ PBQ, = ∠QPB

⇒ PQ = QB = d

Also ∠BRA is ext. ∠ of Δ BQR

∴ ∠QBR = 3α - 2α = α

And ∠BRQ = π – 3 α (linear pair)

Now in ΔBQR, by applying Sine Law, we get

$d/\sin(\pi - 3\alpha) = 3d/4 / \sin\alpha = x/\sin 2\alpha$

$\Rightarrow d/\sin 3\alpha = 3d/4 \sin\alpha = x/\sin^2\alpha$

$\Rightarrow d/3\sin\alpha - 4\sin^3\alpha = 3d/4\sin\alpha = x/2\sin\alpha\cos\alpha$

$\Rightarrow d/3 - 4\sin^2\alpha = 3d/4 = x/2\cos\alpha$ .... (I)

(I) (II) (III)

From eq. (I), I = II

$\Rightarrow d/3 - 4\sin^2\alpha = 3d/4 \Rightarrow 4 = 9 - 12\sin^2\alpha$

$\Rightarrow \sin^2\alpha = 5/12 \Rightarrow \cos^2\alpha = 7/12$

Also from eq. (I) using (II) and (III), we have

$3d/4 = x/2\cos\alpha \Rightarrow 4x^2 = 9d^2\cos^2\alpha$

$x^2 = 9d^2/4 = 7/12 = 21/16\, d^2$ .... (3)

Again from Δ ABR, we have $\sin 3\alpha = h/x$

$\Rightarrow 3\sin\alpha - 4\sin^3\alpha = h/x \Rightarrow \sin\alpha(3 - 4\sin^2\alpha) = h/x$

$\Rightarrow \sin\alpha\,[3 - 4 \times 5/12] = h/x$ (using $\sin^2\alpha = 5/12$)

$\Rightarrow 4/3\sin\alpha = h/x$

Squaring both sides, we get

$16/9\sin^2\alpha = h^2/x^2$ $16/9 \times 2/12 = h^2/x^2$

(again using $\sin^2\alpha = 5/12$)

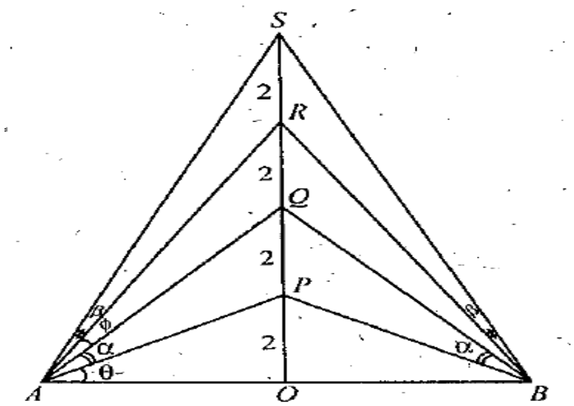
$\Rightarrow h^2 = 4 \times 5/9 \times 3\ x^2 \Rightarrow h^2 = 20/27 \times 21/16\, d^2$

[using value of x^2 from eq. (3)]

$\Rightarrow h^2 = 35/36\, d^2 \Rightarrow 36h^2 = 35d^2$

Which was to be proved.

Sol 17.



Let O be the mid point of AB = 8 m

∴ OA = OB = 4 m

Also OP = 2 m is the initial position of the object which is 2m long. Also we are given, ds/dt = 2t + 1
Integrating we get, $s = t^2 + t + k$ (where s is the distance of top of object from O)

When t = 0, s = OP = 2

∴ k = 2

∴ $s = t^2 + t + 2$ . . . . (1)

For t = 1, s = 4 = OQ ∴ PQ = 2 where PQ is the position of object after 1 sec.

For t = 2, s = 8 = OS but RS = 2 where RS is the position of the object after 2 sec.

∴ OR = OS – RS = 6

Also, OQ = 4

∴ QR = OR – OQ = 6 – 4 = 2

∴ OP = PQ = QR = RS = 2

As per the condition of the question, PQ and RS, the position of the object at t = 1 and t = 2 subtend angles α and β at A and B respectively.

Now let $\angle PAO = \theta$ so that $\tan\theta = 2/4 = 1/2$

Also $\tan(\alpha + \theta) = 4/4 = 1$

Now, $\tan\alpha = \tan[(\alpha + \theta) - \theta]$

$= \tan(\alpha + \theta) - \tan\theta / 1 + \tan(\alpha + \theta)\tan\theta = \frac{1-\frac{1}{2}}{1+\frac{1}{2}} = 1/3$

∴ $\sin\alpha = 1/\sqrt{10}$ and $\cos\alpha = 3/\sqrt{10}$ . . . . (2)

Similarly taking ∠ RAQ = ϕ so that

$\tan\beta = \tan[(\theta+\alpha+\phi+\beta) - (\theta+\alpha+\beta)]$

$= \tan(\theta+\alpha+\phi+\beta) - \tan\theta+\alpha+\beta)/1 + \tan(\theta+\alpha+\phi+\beta).\tan(\theta+\alpha+\beta)$

$= \frac{\frac{8}{4}-\frac{6}{4}}{1+\frac{8}{4}x\frac{6}{4}} = 2\text{-}3/2 \,/\, 1+2(3/2) = 1/8$

$\therefore \sin\beta = 1/\sqrt{65}$ and $\cos\beta = 8/\sqrt{65}$ .... (3)

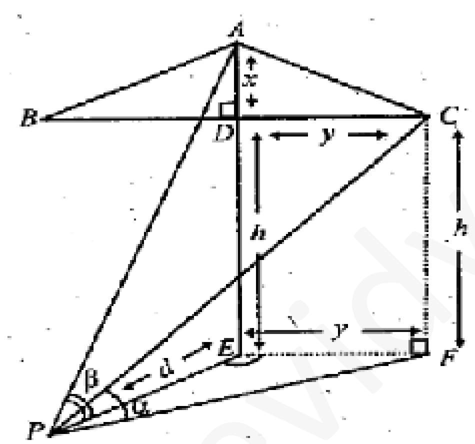
$\therefore \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ Using equations (2) and (3) we get

$\text{Cos}(\alpha-\beta) = 3/\sqrt{10}.\,8/\sqrt{65} + 1/\sqrt{10}.\,1/\sqrt{65}$

$= 25/5\sqrt{2}\sqrt{13} = 5/\sqrt{26}$

Sol 18.

Let ABC be the isosceles triangular sign board with BC horizontal. DE be the pole of height h. Let the man be standing at P such that PE = d



Also ATQ, ∠APE = β

∠CPF = α

Let AD = x be altitude of Δ ABC.

As ΔABC is isosceles with AB = AC

∴ D is mid point of BC.

Hence BC = 2y.

Now in Δ APE,

$\text{Tan}\,\beta = h + x/d \Rightarrow x = d\tan\beta - h$ .... (1)

In ΔCPE, $\tan\alpha = h/PF \Rightarrow \tan\alpha = h/\sqrt{d^2+y^2}$

$\Rightarrow y^2 + d^2 = h^2 \cot^2 \alpha$

$\Rightarrow \quad y = \sqrt{h^2 \cot^2 \alpha - d^2}$ .... (2)

Now area of Δ ABC = 1/2 x BC x AD

= 1/2 x 2y x x = x y

$= (d \tan \beta - h) \sqrt{h^2 \cot^2 \alpha - d^2}$ [Using (1) and (2)]

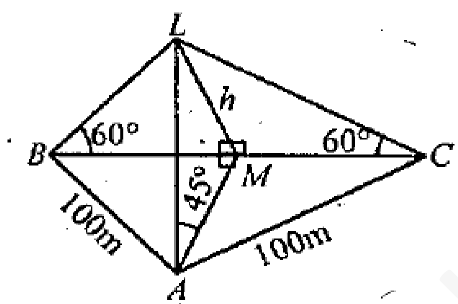
Sol 19.

Let ABC be the triangle region with AB = AC = 100m Let M be the mid point of BC at which tower LM stands.

As ΔABC is isosceles and M is mid pt. of BC

∴ AM ⊥ BC.

Let LM = h be the ht. of tower.



In ΔALM, $\tan 45° = LM/MA \Rightarrow LM = MA$

∴ $MA = h$

Also in ΔBLM, $\tan 60° = LM/BM$

$\Rightarrow \quad \sqrt{3} = h/BM \Rightarrow BM = h/\sqrt{3}$

Now in rt ΔAMB, we have, $AB^2 = AM^2 + BM^2$

$\Rightarrow \quad (100)^2 = h^2 + h^2/3$

$\Rightarrow \quad 4h^2/3 = 10000$

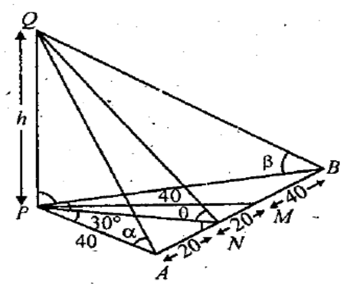
$\Rightarrow \quad h = 50\sqrt{3}$ m.

Sol 20.

Let PQ = h

As A and B are located to the south and east of P respectively,

∴ ∠APB = 90°. M is mid pt of AB. PAM is an equilateral Δ



∴ ∠ APM = 60° :

Also PN ⊥ AB, therefore AN = NM = 20 m

⇒ AP = 40 m

Let angles of elevation of top of the tower from A, N and B be α, θ and β respectively. ATQ, $\tan\theta = 2$

In Δ PQN $\tan\theta = PQ/PN$

⇒ $2 = h/PN \Rightarrow PN = h/2$ .... (1)

Also in ΔAPM, ∠APM = 60° (being equilateral Δ) and PN is altitude ∴ ∠APN = 30°(as in equilateral Δ altitude bisects the vertical angle.

∴ In ΔAPN tan ∠ APN = AN/PN

⇒ $\tan 30° = 20 / h/2$ [Using eq. (1)]

⇒ $h/2\sqrt{3} = 20 \Rightarrow h = 40\sqrt{3}$m.

In ΔAPQ $\tan\alpha = h/AP \Rightarrow \tan\alpha = 40\sqrt{3}/40 = \sqrt{3}$

⇒ α = 60° Also in ΔABQ $\tan\beta = h/PB$ but in rt ΔPNB

$$PB = \sqrt{PN^2 + NB^2} = \sqrt{\left(20\sqrt{3}\right)^2 + (60)^2}$$

∴ $PB = \sqrt{1200 + 3600} = \sqrt{4800} = 40\sqrt{3}$

∴ $\tan\beta = 40\sqrt{3}/40\sqrt{3} \Rightarrow \tan\beta = 1 \Rightarrow \beta = 45°$

Thus h = 40 √3m ; ∠'s of elevation are 60°, 45°

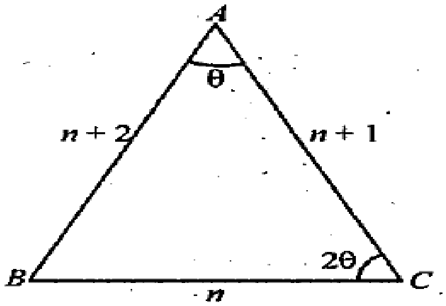
Sol 21.

Let the sides of Δbe n, n + 1, n + 2 where n∈ N.

Let a = n, b = n + 1, c = n + 2

Let the smallest, angle ∠ A = θ then the greatest ∠C = 2θ In ΔABC by applying Sine Law we get,

$\text{Sin}\,\theta/n = \sin 2\theta/n + 2$



⇒ sin θ/n = 2 sin θ cos θ/ n + 2 ⇒ 1/n = 2 cos θ/n + 2 (as sin θ ≠ 0)

⇒ cos θ = n + 2/2n .... (1)

In Δ ABC by Cosine Law, we get

Cos θ = $(n + 1)^2 + (n + 2)^2 - n^2/2(n + 1)(n + 2)$ .... (2)

Comparing the values of cos θ from (1) and (2), we get

$(n + 1)^2 + (n + 2)^2 - n^2/2(n + 1)(n + 2) = n + 2/2n$

⇒ $(n + 2)^2 (n + 1) = n(n + 2)^2 + n(n + 1)^2 - n^3$

⇒ $n (n + 2)^2 (n + 2)^2 = n(n + 2)^2 + n(n + 1)^2 - n^3$

⇒ $n^2 + 4n + 4 = n^3 + 2n^2 + n - n^3$

⇒ $n^2 - 3n - 4 = 0$ ⇒ $(n + 1)(n - 4) = 0$

⇒ n = 4 (as n ≠ - 1)

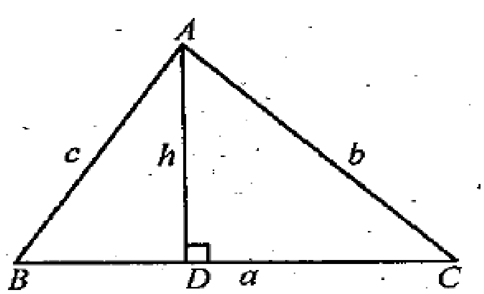
∴ Sides of Δ are 4, 4 + 1, 4 + 2, i.e. 4, 5, 6.

Sol 22.

Given that, In ΔABC, base = a And c/b = r To find altitude, h.

We have, in ΔABD,

h = c sin B = c a sin B/a



= c k sin A sin B/k sin A = c k sin A sin B/sin (B + C)

= c sin A sin B sin (B – C)/sin (B + C) sin (B – C) =c sin A sin B sin (B – C)/$\sin^2 B - \sin^2 C$

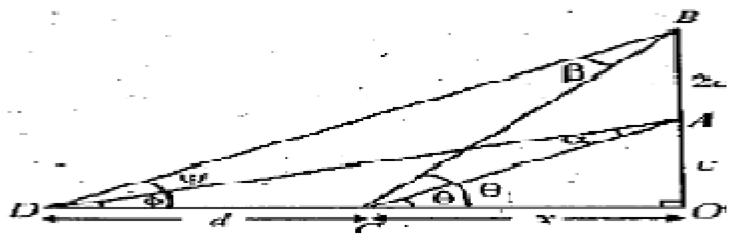
$= \frac{c \cdot \frac{a}{k} \cdot \frac{b}{k} \sin(B-C)}{\frac{b^2}{k^2} - \frac{c^2}{k^2}}$ = abc sin (B – C)/$b^2 - c^2$

= a(c/b) sin (B –C)/1 – $(c/b)^2$ = ar sin (B – C)/1 – $r^2 \leq$ ar/1 – r^2

[∵ sin (B – C) ≤ 1] ∴ h ≤ ar/1 – r^2 Hence Proved.

Sol 23.

Let the man initially be standing at 'A' and 'B' be the position after walking a distance 'c', so total Distance becomes 2c and the objects being observed are at 'C' and 'D'.



Now we have OA = c, AB = 2c

Let CO = x and CD = d

Let ∠CAD = α and ∠CBD = β

∠ACO = θ and ∠ADC = ϕ

∠BCD = ψ and ∠BCO = θ_1

In Δ ACO, tan θ = AO/CO ⇒ tan θ = c/x .... (1)

In Δ ADO, tan ϕ = c/x + d .... (2)

Now, θ = α + ϕ (Using ext. ∠ thm.)

⇒ α = θ - ϕ ⇒ tan α = tan (θ – ϕ) ⇒ = tan θ – tan ϕ/1 + tan θ tan ϕ

= c/x – c/x + d / 1 + c/d. c/x + d (using equations (1) and (2)

⇒ tan α = cx + cd – cx/x^2 + dx + c^2

⇒ $x^2 + c^2$ + xd = cd cot α .... (3)

Again in Δ ADO

tan ψ = 3c/x +d .... (4)

tan θ_1 = 3c/x .... (5)

But θ_1 = ψ + β (by text ∠ thrm)

⇒ β = θ_1 – ψ

$\Rightarrow \tan\beta = \tan(\theta_1 - \psi) = \tan\theta_1 - \tan\psi / 1 + \tan\theta_1 \tan\psi$

$\Rightarrow \tan\beta = \frac{\frac{3c}{x} - \frac{3c}{x+d}}{1 + \frac{3c}{x} \cdot \frac{3c}{x+d}}$ [Using (4) and (5)]

$\Rightarrow \tan\beta = 3cd/x^2 + xd + 9c^2$

$\Rightarrow x^2 + xd + 9c^2 = 3cd \cot\beta$ ....(6)

From (3) and (6), we get

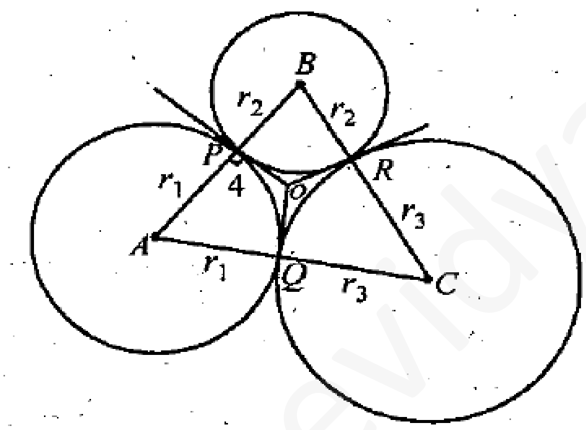
$8c^2 = 3cd \cot\beta - cd \cot\alpha$

$\Rightarrow d = 8c/3 \cot\beta - \cot\alpha$ Hence proved.

Sol 24.

Let us consider circles with centers at A, B and C and with radii r_1, r_2 and r_3 respectively which touch each other externally at P, Q and R. Let the common tangents at P, Q and R meet each other at O. Then OP = OQ = OR = 4 (given)(lengths of tangents from a pt to a circle are equal).

Also $OP \perp AB$, $OQ \perp AC$, $OR \perp BC$.



$\Rightarrow$ O is the in centre of the ΔABC Thus for Δ ABC

$s = (r_1 + r_2) + (r_2 + r_3) + (r_3 + r_1)/2$

i.e. $s = (r_1 + r_2 + r_3)$

$\therefore \Delta = \sqrt{(r_1 + r_2 + r_3)\, r_1 r_2 r_3}$ (Heron's formula)

Now $r = \Delta/s$

NOTE THIS STEP :

$\Rightarrow 4 = \sqrt{(r_1 + r_2 + r_3)\, r_1 r_2 r_3} / r_1 + r_2 + r_3$

$\Rightarrow 4 = \sqrt{r_1 r_2 r_3} / \sqrt{r_1 + r_2 + r_3}$

$\Rightarrow r_1 r_2 r_3 / r_1 + r_2 r_3 = 16/1 \Rightarrow r_1 . r_2 . r_3 : r_1 + r_2 + r_3 = 16 : 1$

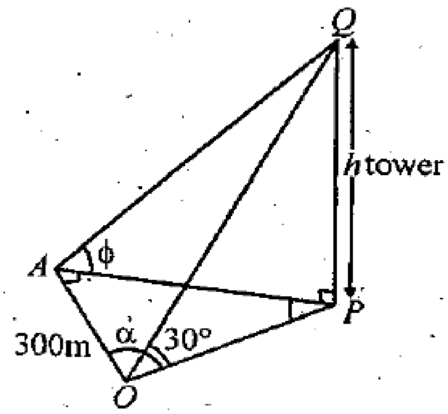
Sol 25.

Let PQ be the tower of height h. A is in the north of O and P is towards east of A.

$\therefore \angle OAP = 90°$

$\angle QOP = 30°$

$\angle QAP = \phi$



$\angle AOP = \alpha$ s. t. $\tan \alpha = 1/\sqrt{2}$

Now in ΔOPQ, $\tan 30° = h/OP \Rightarrow OP = h/\sqrt{3}$ ....(1)

In Δ APQ, $\tan \phi = h/AP \Rightarrow AP = h \cot \phi$ .... (2)

Given that,

$\tan \alpha = 1/\sqrt{2} \Rightarrow \sin \alpha = \tan \alpha/\sqrt{1 + \tan^2 \alpha} = 1/3$

Now in Δ AOP, $\sin \alpha = AP/OP \Rightarrow 1/\sqrt{3} = h \cot \phi/h \sqrt{3}$ [Using (1) and (2)]

$\Rightarrow \cot \phi = 1$

$\Rightarrow \phi = 45°$

Again in ΔOAP, using Pythagoras theorem, we get
$OP^2 = OA^2 + AP^2$

$\Rightarrow 3h^2 = 90000 + h^2 \cot^2 45°$

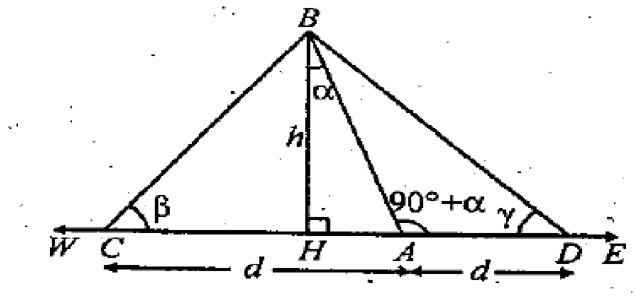
$\Rightarrow h = 150\sqrt{2}$ m

Sol 26.

Let AB be the tower leaning towards west making an angle α with vertical

At C, ∠of elevation of B is β and at D the

∠ of elevation of B is γ, CA = AD = d



When in Δ ABH

⇒ tan α = AH/h ⇒ AH = h tan α .... (1)

In Δ BCH, tan β = h/CH ⇒ CH = h cot β

⇒ d – AH = h cot β

⇒ d = h (tan α + cot β) .... (2)

(Using eq^n (1))

In Δ BDH, tan γ = BH/HD ⇒ h/AH + d

⇒ AH + d = cot γ

⇒ d = h (cot γ – tan α) .... (3)

(Using eq^n (1))

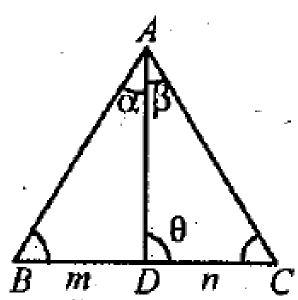
Comparing the values of d from (2) and (3), we get

h(tan α + cot β) = h (cot γ – tan α)

⇒ 2 tan α = cot γ – cot β Hence Proved

ALTERNATE SOLUTION :

KEY CONCEPT:



m : n theorem : In Δ ABC where point D divides BC in the ratio m : n and ∠ ADC = θ

(i) (m + n) cot θ = n cot B - m cot C

(ii) $(m + n) \cot \theta = m \cot \alpha - n - \cot \beta$

In Δ BCD, A divides CD in the ratio 1 : 1 where base ∠'s are β and γ and ∠ BAD = 90° + α

∴ By applying m : n theorem we get

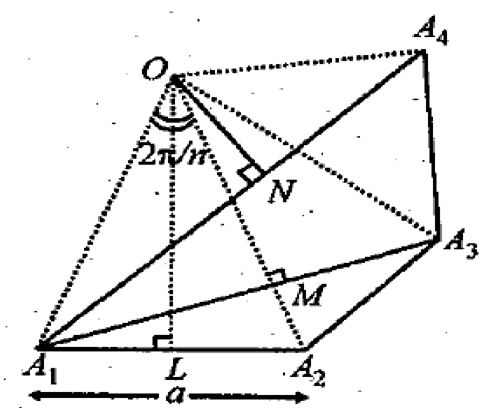
$(1 + 1) \cot (90° + \alpha) = 1. \text{Cot } \beta - 1. \cot \gamma$

$\Rightarrow -2 \tan \alpha = \cot \beta - \cot \gamma \Rightarrow 2 \tan \alpha = \cot \gamma - \cot \beta$ Hence Proved.

<u>**Sol 27.**</u>

Let a be the side of n sided regular polygon

$A_1 A_2 A_3 A_4 \ldots\ldots\ldots\ldots\ldots\ldots A_n$



∴ ∠ Subtended by each side at centre will be

$= 2\pi/n$

Let $OL \perp A_1 A_2$

Then $\angle OLA_1 = 90°$, $\angle A_1 OL = \pi/n$ $(\because OA_1 = OA_2)$

∴ In $\Delta OA_1 L$, $\sin A_1 OL = A_1 L / O A_1$

$\Rightarrow \sin \pi/n = a/2 \,/\, O A_1$

$\Rightarrow O A_1 = a/2 \sin \pi/n$ .... (1)

Again by geometry it can be proved that $OM \perp A_1 A_3$

In $\Delta A_1 M$, $\sin 2\pi/n = A_1 M / O A_1$

$\Rightarrow A_1 M = OA_1 \sin 2\pi/n$

$\Rightarrow A_1 A_3 = 2a \sin 2\pi/n \,/\, 2 \sin \pi/n$ [Using eq^n (1)]

Also if $ON \perp A_1 A_4$, then ON bisects angle

$\angle A_1 OA_4 = 3(2\pi/n)$

$\therefore \angle A_1 ON = 3\pi/n$

$\therefore$ In $\Delta OA_1 N$, $\sin 3\pi/2 = A_1 N/O A_1$

$\Rightarrow A_1 N = O A_1 \sin 3\pi/h/n$

$\Rightarrow A_1 N_4 = 2A_1 N = 2a \sin 3\pi/n / 2 \sin \pi/n$

But given that

$1/A_1 A_2 = 1/A_1 A_3 + 1/A_1 A_4$

$\Rightarrow 1/a = 1/a \sin 2\pi/n / \sin \pi/n \quad + \quad 1/a \sin 3\pi/n / \sin \pi/n$

$\Rightarrow \sin 3\pi/n \sin 2\pi/n = (\sin 3\pi/n + \sin 2\pi/n) \sin \pi/n$

$\Rightarrow 2 \sin 3\pi/n \sin 2\pi/n = 2 \sin 3\pi/n + 2 \sin 2\pi/n \sin \pi/n$

$\Rightarrow \cos \pi/n - \cos 5\pi/n = \cos 2\pi/n - \cos 4\pi/n + \cos \pi/n - \cos 3\pi/n$

$\Rightarrow \cos 2\pi/n + \cos 5\pi/n = \cos 4\pi/n + \cos 3\pi/n$

$\Rightarrow 2 \cos 7\pi/2n \ \cos 3\pi/2n = 2 \cos 7\pi/2n \cos \pi/2n$

$\Rightarrow \cos 7\pi/2n (\cos 3\pi/2n - \cos \pi/2n) = 0$

$\Rightarrow \cos 7\pi/2n \,.\, 2 \sin 2\pi/n \sin \pi/n = 0$

$\Rightarrow \quad \cos 7\pi/2n = 0$

Or $\sin 2\pi/n = 0$ or $\sin \pi/n = 0$

$\Rightarrow 7\pi/2n = (2k + 1)\pi/2$ or $2\pi/n = k\pi/n$ or $\pi/n = p\pi$

$\Rightarrow n = 7/2k + 1$ or $n = 2/k$ or $n = 1/k$

But n should be a +ve integer being no. of sides and $n \geq 4$ (four vertices being considered in the question)

$\therefore$ the only possibility is $n = 7/2k + 1$ for $k = 0$

$\therefore n = 7$

Sol 28.

(I) a, b, c and Δ are rational.

$\Rightarrow s = a + b + c/2$ is also rational

$\Rightarrow \tan B/2 = \sqrt{}(s - a)(s - c)/s(s - b) = \Delta/s(s - b)$ is also rational

And $\tan C/2 = \sqrt{}(s - a)(s - b)/s(s - c) = \Delta/s(s - c)$ is also rational

Hence (I) $\Rightarrow$ (II).

(II) a, tan B/2, tan C/2 are rational.

⇒ sin B = 2tan B/2 / a + $\tan^2$ B/2

And sin C = 2 tan C/2 / 1 + $\tan^2$ C/2 are rational

And tan A/2 = tan [90° - (B/2 + C/2)] = cot (B/2 + C/2)

= 1/tan (B/2 + C/2) = 1 – tan B/2 tan C/2 / tan B/2 + tan C/2 is rational

∴ sin A = 2tan A/2 / 1 + $\tan^2$ A/2 is rational.

Hence (II) ⇒ (III)

(III) a, sin A, sin B, sin C are rational.

But a/sin A = 2R

⇒ R is rational

∴ b = 2R sin B, c = 2R sin C are rational.

∴ Δ = 1/2 bc sin A is rational

Hence (III) ⇒ (I).

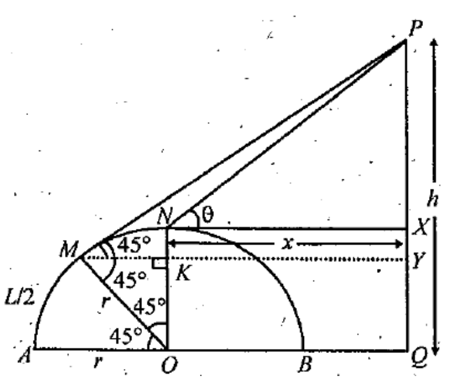
Sol 29.

Let AMNB be the semicircular arc of length 2L, and PQ be the vertical tower so that A, B, Q are in the same horizontal line.

Let M be the pt. on arc s. t. AB = L/2 then as AM = 1/4 AB we should have ∠AOM = 45°, As at M the man just sees the top most pt P of the tower, tangent through M must pass through P and hence ∠OMP = 90° and then by simple geometry we get ∠PMY = 45°.

Also N is the top most pt. of arc AB, hence ON must be vertical.

∴ ON = r = XQ



∴ PX = h – r, where h is ht of tower PQ, and OK = YQ = OM cos 45° = r/√2

Similarly, MK = OM sin 45° = r√2

$\therefore$ In Δ PMY we get tan 45° = PY/MY

$\Rightarrow$ PY = MY

$\Rightarrow$ h – QY = MK + KY

$\Rightarrow$ h – OK = r/√2 + x

$\Rightarrow$ h – r/√2 + r/√2 + x

$\Rightarrow$ x = h – r √2 .... (1)

In Δ PNX, tan θ = PX/NX

$\Rightarrow$ tan θ = (h – r)/x

$\Rightarrow$ x = (h – r) cot θ .... (2)

Comparing the values of x from (1)and (2), we get

h – r √2 = h cot θ – r cot θ

h = r (√2 – cot θ)/(1 – cot θ)

But are length = 2L = π r ⇒ r = 2L/π

$\therefore$ h = 2L/π [√2 – cot θ/1 – cot θ]

<u>Sol 30.</u>

Given that A, B, C, are three ∠'s of a Δ therefore

A + B + C = π Also A = π/4 ⇒ B + C = 3 π/4 ⇒ 0 < B, C < 3π/4

Now tan B tan C = P

$\Rightarrow$ sin B sin C/cos B cos C = p/1

Applying componendo and dividendo, we get

Sin B sin C + cos B cos C/cos B cos C – sin B sin C = 1+p/1-p

$\Rightarrow$ cos (B – C)/cos (B + C) = 1 + p/1 – p

$\Rightarrow$ cos (B – C) = 1 + p/1 – p (-1/√2) .... (1) [∵ B + C = 3 π/4]

Now, as B and C can vary from 0 to 3π/4

$\therefore$ 0 ≤ B – C < 3π/4

$\Rightarrow$ 1/√2 < cos (B – C) ≤ 1

From eq" (1) substituting the value of cos (B – C), we get

$-1/\sqrt{2} < 1 + p/\sqrt{2}(p - 1) \geq 1$

$\Rightarrow -1/\sqrt{2} < 1 + p/\sqrt{2}(p - 1)$ and $1 + p/\sqrt{2}(p - 1) \leq 1$

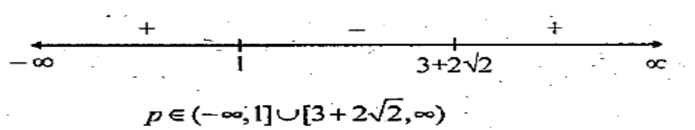
$\Rightarrow 0 < 1 + \quad p + 1/p - 1$ and $(p + 1) - \sqrt{2}\,(p - 1)/\sqrt{2}\,(p - 1) \quad \leq 0$

$\Rightarrow 2p/p - 1 > 0$ and $\quad p + 1 - \sqrt{2}\,p + \sqrt{2} \,/\, \sqrt{2}(p - 1) \leq 0$

$\Rightarrow p\,(p - 1) > 0$ and $(1 - \sqrt{2})\,p + (\sqrt{2} + 1)/(p - 1) \leq 0$

$\Rightarrow p \in (-\infty, 0) \cup (1, \infty)$, and $-p + (\sqrt{2} + 1)^2/(p - 1) \leq 0$

$\Rightarrow [p - (3 + 2\sqrt{2})]\,[p - 1] \geq 0$



Combining the two cases, we get

$p \in (-\infty, 0) \cup [3 + 2\sqrt{2}, \infty)$.

Sol 31.

Let A, B, C be the projections of the pts.

P, Q and M on the ground.

ATQ, $\angle POA = 60°$

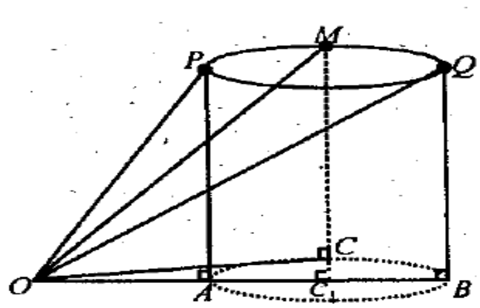
$\angle QOB = 30°$

$\angle MOC = \theta$

Let h be the ht of circle from ground, then

$AP = CM = BQ = h$

Let $OA = x$ and $AB = d$(diameter of the projection of the circle on ground with C_1 as centre).



Now in ΔPOA, $\tan 60° = h/x \Rightarrow x = h/\sqrt{3}$ .... (1)

In ΔQBO, $\tan 30° = h/x + d \Rightarrow x + d = h\sqrt{3}$

$\Rightarrow d = h\sqrt{3} - h/\sqrt{3} = 2h/\sqrt{3}$ . . . . (2)

In ΔOMC, $\tan\theta = h/OC$

$\Rightarrow \tan^2\theta = h^2/OC^2 = h^2/OC^2{}_1 + C_1C^2 = h^2/(x + d/2)^2 + (d/2)^2$

$= \frac{h^2}{\left(\frac{h}{\sqrt{3}}+\frac{h}{\sqrt{3}}\right)^2+\left(\frac{h}{\sqrt{3}}\right)^2}$ [Using (1) and (2)

$= \frac{h^2}{\frac{4h^2}{3}+\frac{h^2}{3}} = 3/5$

Sol 32.

Let ABC is an equilateral Δ then

$A = B = C = 60°$

$\Rightarrow \tan A + \tan B \tan C = 3\sqrt{5}$

Conversely, suppose

$\tan A + \tan B + \tan C = 3\sqrt{3}$ . . . . (1)

Now using A. M. ≥ G. M. (equality occurs when no's are equal)

For tan A, tan B, tan C, we get

$\tan A + \tan B + \tan C/3 \geq (\tan A \tan B \tan C)^{1/3}$

NOTE THIS STEP :

But in any ΔABC, know that

$\tan A + \tan B \tan C = \tan A \tan B \tan C$

∴ Last inequality becomes

$\tan A + \tan B + \tan C/3 \geq (\tan A + \tan B + \tan C)^{1/3}$

$\Rightarrow (\tan A + \tan B + \tan C)^{2/3} \geq 3$

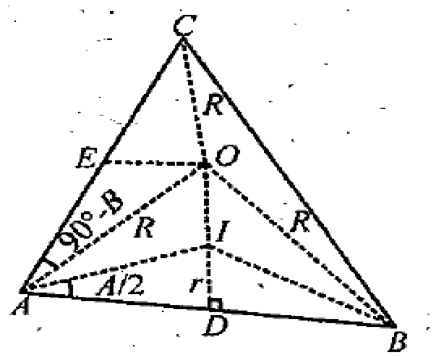
$\Rightarrow \tan A + \tan B + \tan C \geq 3\sqrt{3}$

Where equality occurs when tan A, tan B, tan C are equal, i.e. A = B = C

⇒ ΔABC is equilateral.

Sol 33.

In ΔABC, O and I are circumcentre and indenter of Δ respectively and R and r are the respective radii of circum circle and in circle.



To prove $(IO)^2 = R^2 - 2Rr$

First of all we will find IO. Using cosine law in ΔAOI

$\cos \angle OAI = OA^2 + AI^2 - OI^2/2. OA. AI$ ....(1)

where OA = R

In Δ AID, sin A/2 = r/AI

AI = r/sin A/2

= 4R sin B/2 sin C/2 [Using r = 4R sin A/2 sin B/2 sin C/2]

Also, ∠OAI = ∠IAE - ∠OAE

= A/2 – (90° - ∠AOE)

= A/2 - 90° + 1/2 ∠AOC

= A/2 - 90° + 1/2. 2B (∵ O is circumcentre ∴ ∠AOC = 2∠B)

= A/2 + B – A + B + C/2

= B – C/2

Substituting all these values in equation (1) we get

$\text{Cos } (B - C)/2 = R^2 + 16 R^2 \sin^2 B/2 \sin^2 c/2 - OI^2 \mid 2.R.4 R \sin C/2$

$\Rightarrow OI^2 = R^2 + 16R^2 \sin^2 B/2 \sin^2 C/2 - 8R^2 \sin B/2 \sin C/2 \cos B - C/2$

$= R^2 [1 + \sin B/2 \sin C/2\{2 \sin B/2 \sin C/2 - \cos B - C/2 \}]$

$= R^2 [1 + 8 \sin B/2 \sin C/2 \{2 \sin B/2 \sin C/2 - \cos B/2 \cos C/2 - \sin B/2 \sin C/2\}]$

$= R^2 [1 + 8 \sin C/2 \{ \sin B/2 \sin C/2 - \cos B/2 \cos C/2]$

$= R^2 [1\text{-}8 \sin B/2 \sin C/2 \cos B + C/2]$

$= R^2 [1 - 8 \sin A/2 \sin B/2 \sin C/2]$

$= R^2 - 2R. 4R \sin A/2 \sin B/2 \sin C/2$

= $R^2 - 2Rr$. Hence Proved

Again if Δ OIB is right ∠ed Δ then

⇒ $OB^2 = OI^2 + IB^2$

⇒ $R^2 = R^2 - 2Rr + r^2/\sin^2 B/2$

NOTE THIS STEP:

[∵ In Δ IBD $\sin B/2 = r/IB$]

⇒ $2R \sin^2 B/2 = r$

⇒ $2abc/4\Delta\ (s - a)\ (s - c)/ac = \Delta/s$

⇒ $b(s - a)\ (s - c) = 2\ (s - a)(s - b)(s - c)$

⇒ $b = 2s - 2b$ ⇒ $b = a + b - c$

⇒ $a + c = 2b$ ⇔ a, b, c are in A. P.

⇒ b is A. M> between a and c. Hence Proved.s

Sol 34.

Let $MN = r_3 = MP = MQ$, $ID = r$

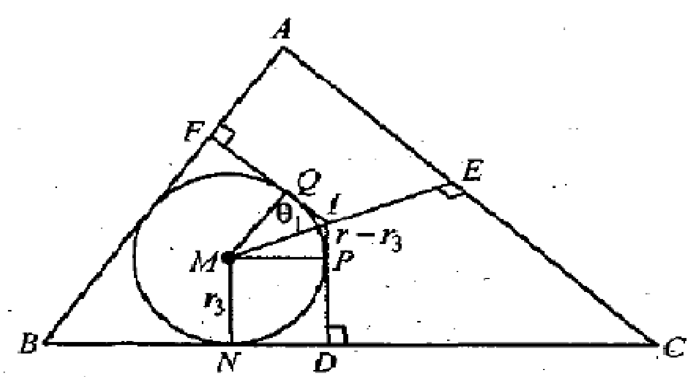
⇒ $IP = r - r_3$

Clearly IP and IQ are tangents to circle with centre M.

∴ IM must be the ∠ bisector of ∠ PIQ

∴ $\angle PIM = \angle QIM = \theta_1$

Also from Δ IPM, $\tan \theta_1 = r_3/r - r_3 = MP/IP$



Here DI = r

Similarly, in other quadrilaterals, we get

$\tan \theta_2 = r_2/r - r_2$ and $\tan \theta_3 = r_1/r - r_1$

Also $2\theta_1 + 2\theta_2 + 2\theta_3 = 2\pi \Rightarrow \theta_1 + \theta_2 + \theta_3 = \pi$

$\Rightarrow \tan\theta_1 \tan\theta_2 + \tan\theta_1\, \theta_3 = \tan\theta_1 . \tan\theta_{2.} \tan\theta_3$

NOTE THIS STEP:

$= r_1/r - r_1 + r_2/r - r_2 + r/r - r_3 = r_1 r_2 r_3/(r - r_1)(r - r_2)(r - r_3)$

Sol 35.

We know, $\Delta = \sqrt{s}\,(s - a)(s - b)(s - c)$

$= \sqrt{s/8}\;(b + c - a)(c + a - b)(a + b - c)$

Since sum of two sides is always greater than third side;

$\therefore b + c - a, c + a - b, a + b - c > 0$

$\Rightarrow (s - a)(s - b)(s - c) > 0$

Let $s - a = x. s - b = y, s - c = z$

Now, $x + y = 2s - a - b = c$

Similarly, $y + z = a$

And $z + x = b$

Since AM $\geq$ GM

$\Rightarrow x + y/2 \geq \sqrt{xy} \Rightarrow 2\sqrt{xy} \leq a \quad y + z/2 \geq \sqrt{yz} \Rightarrow 2\sqrt{yz} \leq b \quad z + x/2 \geq \sqrt{xz} \Rightarrow 2\sqrt{xz} \leq c \therefore 8xyz \leq abc$

$\Rightarrow (s - a)(s - b)(s - c) \leq 1/8\ abc$

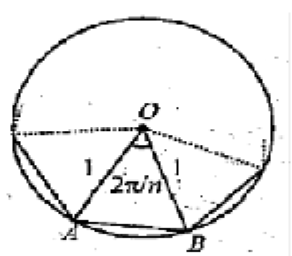
$\Rightarrow s(s - a)(s - b)(s - c) \leq sabc/8$

$\Rightarrow \leq 1/16\,(a + b + c)\,abc \Rightarrow \Delta \leq 1/4\,\sqrt{abc\,(a + B + c)}$

And equality holds when $x = y = z \Rightarrow a = b = c$

Sol 36.

Let OAB be one triangle out of n on a n sided polygon inscribed in a circle of radius 1.



Then $\angle AOB = 2\pi/n$

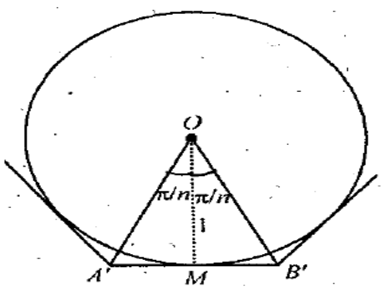
OA = OB = l

∴ Using Area of isosceles Δ with vertical ∠θ and equal

Sided as $r = 1/2 r^2 \sin\theta = 1/2 \sin 2\pi/n$

∴ $I_n = n/2 \sin 2\pi/n$ . . . . (1)

Further consider the n sided polygon subscribing on the circle.



A' M' B' is the tangent of the circle at M.

⇒ A' M B' ⊥ OM

⇒ A' MO is right angled triangle, right angle at M.

A' M = tan π/n

Area of ΔA' MO = 1/2 x 1 x tan π/n

∴ Area of ΔA' B' O = tan π/n

So, $O_n = n \tan \pi/n$ . . . . (2)

Now, we have to prove

$$I_n = \frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2l_n}{n}\right)^2}\right)$$

Or $2I_n/O_n - 1 = \sqrt{1-\left(\frac{2l_n}{n}\right)^2}$

LHS = $2L_n/O_n - 1 = n \sin 2\pi/n \,/\, n \tan \pi/n - 1$ (From (1) and (2))

$= 2\cos^2 \pi/n - 1 = \cos 2\pi/n$

RHS = $\sqrt{1-\left(\frac{2l_n}{n}\right)^2}$ = =$\sqrt{1 - \sin^2 2\pi/n}$ (From (1))

= cos (2π/n) Hence Proved.