CBSE 2013 ALL INDIA EXAMINATION
[Set 1 With Solutions]

Max. Marks : 100 **Time Allowed : 3 Hours**

SECTION – A

Q01. Write the principal value of $\tan^{-1}(\sqrt{3})-\cot^{-1}(-\sqrt{3})$.

Sol. $\tan^{-1}(\sqrt{3})-\cot^{-1}(-\sqrt{3}) = \tan^{-1}\tan\frac{\pi}{3}-\cot^{-1}\left(-\cot\frac{\pi}{6}\right)$

$\Rightarrow \quad = \frac{\pi}{3}-\cot^{-1}\cot\left(\pi-\frac{\pi}{6}\right)$ [Range of $\tan^{-1}:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\cot^{-1}:\ ]0,\pi[$

$\Rightarrow \quad = \frac{\pi}{3}-\cot^{-1}\cot\frac{5\pi}{6} = \frac{\pi}{3}-\frac{5\pi}{6} = -\frac{\pi}{2}$.

Q02. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$.

Sol. Let $Y = \tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right] = \tan^{-1}\left[2\sin\left(2\cos^{-1}\cos\frac{\pi}{6}\right)\right]$

$= \tan^{-1}\left[2\sin\left(2\times\frac{\pi}{6}\right)\right] = \tan^{-1}\left[2\sin\left(\frac{\pi}{3}\right)\right] = \tan^{-1}\left[2\times\frac{\sqrt{3}}{2}\right]$

$= \tan^{-1}\sqrt{3} = \tan^{-1}\tan\frac{\pi}{3}$

$\therefore Y = \frac{\pi}{3}$.

Q03. For what value of x, is the matrix $\begin{bmatrix}0 & 1 & -2\\ -1 & 0 & 3\\ x & -3 & 0\end{bmatrix}$ a skew-symmetric matrix?

Sol. Let $A = \begin{bmatrix}0 & 1 & -2\\ -1 & 0 & 3\\ x & -3 & 0\end{bmatrix}$. Since A is skew-symmetric matrix, which means $A = -A^T$.

$\therefore \begin{bmatrix}0 & 1 & -2\\ -1 & 0 & 3\\ x & -3 & 0\end{bmatrix} = -\begin{bmatrix}0 & 1 & -2\\ -1 & 0 & 3\\ x & -3 & 0\end{bmatrix}^T \Rightarrow \begin{bmatrix}0 & 1 & -2\\ -1 & 0 & 3\\ x & -3 & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & -x\\ -1 & 0 & 3\\ 2 & -3 & 0\end{bmatrix}$

By equality of matrices, $-x=-2 \Rightarrow x=2$.

Q04. If matrix $A = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ and $A^2 = kA$, then write the value of k.

Sol. Given $A = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ and $A^2 = kA \Rightarrow A.A = kA$.

$\therefore \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1+1 & -1-1\\ -1-1 & 1+1\end{bmatrix} = k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix} = k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \Rightarrow 2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = k\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \Rightarrow 2A = kA$

$\Rightarrow 2AA^{-1} = kAA^{-1}$ [Post multiplying both the sides by A^{-1}]

$\Rightarrow 2I = kI \quad \Rightarrow k=2$.

Q05. Write the differential equation representing the family of curves $y = mx$, where m is an arbitrary constant.

Sol. We have $y = mx$ ...(i)

Differentiating w.r.t. x both the sides, $\frac{dy}{dx} = \frac{d}{dx}(mx) \Rightarrow \frac{dy}{dx} = m$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$ [By using eq. (i)

This is the required differential equation.

Q06. If A_{ij} is the cofactor of the element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ then, write the value of $a_{32}.A_{32}$.

Sol. We have $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$.

$\therefore \ a_{32}.A_{32} = 5 \times (-1)^{3+2} \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = -5(2 \times 4 - 6 \times 5)$

i.e, $\therefore \ a_{32}.A_{32} = 110$.

Q07. P and Q are two points with position vectors $3\vec{a} - 2\vec{b}$ and $\vec{a} + \vec{b}$ respectively. Write the position vector of a point R which divides the line segment PQ in the ratio 2:1 externally.

Sol. Given $\overrightarrow{OP} = 3\vec{a} - 2\vec{b}$, $\overrightarrow{OQ} = \vec{a} + \vec{b}$. Also R divides PQ in 2:1 externally.

$\therefore \overrightarrow{OR} = \frac{2(\overrightarrow{OQ}) - 1(\overrightarrow{OP})}{2-1} = \frac{2(\vec{a} + \vec{b}) - 1(3\vec{a} - 2\vec{b})}{1}$

i.e., $\overrightarrow{OR} = 4\vec{b} - \vec{a}$.

Q08. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 15$.

Sol. Given $(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 15$ $\Rightarrow \vec{x}.\vec{x} + \vec{x}.\vec{a} - \vec{a}.\vec{x} - \vec{a}.\vec{a} = 15$ $\Rightarrow |\vec{x}|^2 + \vec{x}.\vec{a} - \vec{x}.\vec{a} - |\vec{a}|^2 = 15$

$\Rightarrow |\vec{x}|^2 - (1)^2 = 15$ $\Rightarrow |\vec{x}|^2 = 15 + 1$

$\Rightarrow |\vec{x}| = \pm\sqrt{16}$ $\therefore |\vec{x}| = 4$.

Q09. Find the length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$.

Sol. The length of the perpendicular drawn from the origin (0, 0) to the plane $2x - 3y + 6z + 21 = 0$ is :

$$p = \frac{|2(0) - 3(0) + 6(0) + 21|}{\sqrt{2^2 + (-3)^2 + 6^2}} \quad \text{[Using } p = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} units$$

i.e., $p = \frac{21}{\sqrt{4 + 9 + 36}} units = 3 units$.

Q10. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in ₹) received from the sale of x units of a product is given by R(x) = $3x^2 + 36x + 5$, find the marginal revenue, when $x = 5$, and write which value does the question indicate.

Sol. We have total revenue R(x) = $3x^2 + 36x + 5$

So, Marginal Revenue = Rate of change of total revenue function

i.e., $= \frac{d}{dx}(3x^2 + 36x + 5) = 6x + 36$

$\therefore$ Marginal Revenue when $x = 5 = 6(5) + 36 = 66$.

Value indicated : More is the revenue generated by the firm, more will be the money spent for the welfare of employees.

SECTION – B

Q11. Consider $f : R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that *f* is invertible with the inverse f^{-1} of *f* given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

Sol. Given $f : R_+ \to [4, \infty)$ defined as $f(x) = x^2 + 4$.

For one-one : Let $x_1, x_2 \in R_+$. We have $f(x_1) = f(x_2)$.

i.e., $x_1^2 + 4 = x_2^2 + 4 \quad \Rightarrow x_1^2 - x_2^2 = 0 \quad \Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$

$\Rightarrow (x_1 - x_2) = 0 \quad \Rightarrow x_1 = x_2$ [$\because x_1 + x_2 \neq 0$ as $x_1, x_2 \in R_+$

So, *f* is one-one.

For onto : Let $y \in [4, \infty)$ and $y = f(x)$.

$\therefore\ y = x^2 + 4 \quad \Rightarrow y - 4 = x^2 \quad \Rightarrow x = \pm\sqrt{y-4}$

$\Rightarrow x = \sqrt{y-4}$ [$\because x \in R_+$

Clearly, $x \in R_+$ for all $y \in [4, \infty)$.

Thus, for each $y \in [4, \infty)$ there exists $x = \sqrt{y-4} \in R_+$.

$\therefore$ *f* is onto function.

Since *f* is one-one and onto function, so it is bijective function and hence it is invertible.

Also $f^{-1} : [4, \infty) \to R_+$ is given as $f^{-1}(y) = \sqrt{y-4}$.

Q12. Show that : $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$. **OR** Solve : $\cos(\tan^{-1} x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$.

Sol. **LHS:** Let Y = $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$

Put $\sin^{-1}\frac{3}{4} = x \Rightarrow \sin x = \frac{3}{4} \Rightarrow \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{16}} = \frac{\sqrt{7}}{4}$ …(i)

Now Y = $\tan\left(\frac{x}{2}\right) = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \frac{2\sin\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)}{2\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)} = \frac{2\sin^2\left(\frac{x}{2}\right)}{\sin x}$

$\Rightarrow \quad = \frac{1-\cos x}{\sin x} = \frac{1-\frac{\sqrt{7}}{4}}{\frac{3}{4}}$ [By (i)]

$\Rightarrow \quad = \frac{4-\sqrt{7}}{3} =$ **RHS.** **[Hence Proved.**

OR

We have $\cos(\tan^{-1} x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

$\Rightarrow \cos\left[\frac{\pi}{2} - \cot^{-1} x\right] = \sin\left(\cot^{-1}\frac{3}{4}\right)$ [$\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

$\Rightarrow \sin(\cot^{-1} x) = \sin\left(\cot^{-1}\frac{3}{4}\right) \quad \Rightarrow \cot^{-1} x = \cot^{-1}\frac{3}{4}$

$\therefore\ x = \frac{3}{4}$.

Q13. Using properties of determinants, prove that : $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y)$.

Sol. **LHS:** Let $\Delta = \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

$\Rightarrow \quad = \begin{vmatrix} 3x+3y & x+y & x+2y \\ 3x+3y & x & x+y \\ 3x+3y & x+2y & x \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 + C_3$

$\Rightarrow \quad = (3x+3y)\begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}$ [Taking $3x+3y$ common from C_1

$\Rightarrow \quad = (3x+3y)\begin{vmatrix} 1 & x+y & x+2y \\ 0 & -y & -y \\ 0 & y & -2y \end{vmatrix}$ [Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$\Rightarrow \quad = (3x+3y)(y)^2\begin{vmatrix} 1 & x+y & x+2y \\ 0 & -1 & -1 \\ 0 & 1 & -2 \end{vmatrix}$ [Taking y common from R_2 & R_3

$\Rightarrow \quad = y^2(3x+3y)\left[1 \times \begin{vmatrix} -1 & -1 \\ 1 & -2 \end{vmatrix}\right]$ [Expanding along C_1

$\Rightarrow \quad = y^2(3x+3y)\left[1 \times (2+1)\right]$

$\Rightarrow \quad = 9y^2(x+y) =$ **RHS.** **[Hence Proved.**

Q14. If $y^x = e^{y-x}$, prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$.

Sol. Given $y^x = e^{y-x}$

Taking log on both the sides, $\log(y^x) = \log(e^{y-x})$ $\Rightarrow x\log y = (y-x)\log e = y - x$ [$\because \log e = 1$

$\Rightarrow x = \frac{y}{1+\log y}$ [Differentiating w.r.t. y both the sides

$\Rightarrow \frac{dx}{dy} = \frac{(1+\log y)\frac{dy}{dy} - y\frac{d}{dy}(1+\log y)}{(1+\log y)^2}$ $\Rightarrow \frac{dx}{dy} = \frac{(1+\log y).1 - y\left(0+\frac{1}{y}\right)}{(1+\log y)^2}$

$\Rightarrow \frac{dx}{dy} = \frac{\log y}{(1+\log y)^2}$ $\therefore \frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$ **[Hence Proved.**

Q15. Differentiate $\sin^{-1}\left(\frac{2^{x+1}.\ 3^x}{1+(36)^x}\right)$ with respect to x.

Sol. Let $y = \sin^{-1}\left(\frac{2^{x+1}.\ 3^x}{1+(36)^x}\right)$ $\Rightarrow y = \sin^{-1}\left(\frac{2\times 2^x.\ 3^x}{1+(36)^x}\right)$ $\Rightarrow y = \sin^{-1}\left(\frac{2\times 6^x}{1+(6^x)^2}\right)$

$\Rightarrow y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$, where $6^x = \tan\theta \Rightarrow \theta = \tan^{-1} 6^x$

$\Rightarrow y = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1} 6^x$ [Diff. w.r.t. x both the sides

$\therefore \frac{dy}{dx} = 2\times\frac{d}{dx}(\tan^{-1} 6^x) = 2\times\frac{1}{1+(6^x)^2}\frac{d}{dx}(6^x)$ i.e., $\frac{dy}{dx} = \frac{2}{1+36x^2}\left[6^x(\log 6)\right]$

Hence, $\frac{dy}{dx}=\frac{2.2^x.3^x}{1+36x^2}\left[\log 6\right]=\frac{2^{x+1}.3^x(\log 6)}{1+36x^2}$.

Q16. Find the value of *k*, for which $f(x)=\begin{cases}\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, \text{ if } -1\le x<0\\ \frac{2x+1}{x-1}, \text{ if } 0\le x<1\end{cases}$ is continuous at $x=0$.

OR If $x=a\cos^3\theta$ and $y=a\sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta=\frac{\pi}{6}$.

Sol. Since *f* is continuous at $x=0$ so, $\lim_{x\to 0}f(x)=f(0)$ ...(i)

Now, $f(0)=\frac{2(0)+1}{(0)-1}=-1$ ...(ii)

And, LHL (at $x=0$) : $\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}$

$$=\lim_{x\to 0^-}\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}\times\frac{\sqrt{1+kx}+\sqrt{1-kx}}{\sqrt{1+kx}+\sqrt{1-kx}}$$

$$=\lim_{x\to 0^-}\frac{(1+kx)-(1-kx)}{x}\times\frac{1}{\sqrt{1+kx}+\sqrt{1-kx}}$$

$$=\lim_{x\to 0^-}\frac{2k}{\sqrt{1+kx}+\sqrt{1-kx}}=\frac{2k}{2}=k \quad \text{...(iii)}$$

By (i), (ii) and (iii), we have : $k=-1$.

OR

We have $x=a\cos^3\theta$ and $y=a\sin^3\theta$

Differentiating w.r.t. θ, $\frac{dx}{d\theta}=\frac{d}{d\theta}(a\cos^3\theta)=-3a\cos^2\theta\sin\theta$ ...(i)

And, $\frac{dy}{d\theta}=\frac{d}{d\theta}(a\sin^3\theta)=3a\sin^2\theta\cos\theta$ ...(ii)

So, $\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}=3a\sin^2\theta\cos\theta\times\frac{1}{(-3a\cos^2\theta\sin\theta)}=-\tan\theta$ [Using (i) & (ii)

Now, $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(-\tan\theta)=-\sec^2\theta\times\frac{d\theta}{dx}$ [Diff. w.r.t. *x* both the sides

$\frac{d^2y}{dx^2}=-\sec^2\theta\times\frac{1}{(-3a\cos^2\theta\sin\theta)}$ [By using (i)

$\therefore \frac{d^2y}{dx^2}=\frac{1}{3a}\times\frac{\sec^4\theta}{\sin\theta}$

So, $\therefore \left.\frac{d^2y}{dx^2}\right]_{at\ \theta=\frac{\pi}{6}}=\frac{1}{3a}\times\frac{\sec^4\frac{\pi}{6}}{\sin\frac{\pi}{6}}=\frac{1}{3a}\times\frac{\left(\frac{2}{\sqrt{3}}\right)^4}{\frac{1}{2}}=\frac{1}{3a}\times\frac{32}{9}=\frac{32}{27a}$.

Q17. Evaluate: $\int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx$. **OR** Evaluate: $\int\frac{x+2}{\sqrt{x^2+2x+3}}dx$.

Sol. Let $I=\int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx=\int\frac{(2\cos^2x-1)-(2\cos^2\alpha-1)}{\cos x-\cos\alpha}dx$

$\Rightarrow \quad = \int \dfrac{2(\cos^2 x - \cos^2 \alpha)}{\cos x - \cos \alpha} dx \quad \Rightarrow \quad = \int \dfrac{2(\cos x - \cos \alpha)(\cos x + \cos \alpha)}{\cos x - \cos \alpha} dx$

$\Rightarrow \quad = 2\int (\cos x + \cos \alpha)\, dx = 2(\sin x + x \cos \alpha) + \text{C}$.

OR

Let $\text{I} = \int \dfrac{x+2}{\sqrt{x^2+2x+3}} dx \quad \Rightarrow \quad = \dfrac{1}{2}\int \dfrac{2x+4}{\sqrt{x^2+2x+3}} dx$

$\Rightarrow \quad = \dfrac{1}{2}\int \dfrac{2x+2}{\sqrt{x^2+2x+3}} dx + \int \dfrac{1}{\sqrt{x^2+2x+3}} dx$

Put $x^2+2x+3 = t$ in 1st integral so, $(2x+2)dx = dt$

$\therefore \text{I} = \dfrac{1}{2}\int \dfrac{dt}{\sqrt{t}} + \int \dfrac{1}{\sqrt{x^2+2x+1+2}} dx \quad \Rightarrow \quad = \dfrac{1}{2} \times 2\sqrt{t} + \int \dfrac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}} dx$

i.e., $\text{I} = \sqrt{x^2+2x+3} + \log\left|x+1+\sqrt{(x+1)^2+(\sqrt{2})^2}\right| + \text{C}$

$\therefore \text{I} = \sqrt{x^2+2x+3} + \log\left|x+1+\sqrt{x^2+2x+3}\right| + \text{C}$.

Q18. Evaluate : $\int \dfrac{dx}{x(x^5+3)}$.

Sol. Let $\text{I} = \int \dfrac{dx}{x(x^5+3)} \quad \Rightarrow \quad = \dfrac{1}{5}\int \dfrac{5x^4}{x^5(x^5+3)} dx$

Put $x^5+3 = t$ so, $5x^4 dx = dt$

$\therefore \text{I} = \dfrac{1}{5}\int \dfrac{dt}{(t-3)t} \quad \Rightarrow \quad = \dfrac{1}{5}\int \dfrac{1}{3} \times \left[\dfrac{1}{t-3} - \dfrac{1}{t}\right] dt$

i.e., $\text{I} = \dfrac{1}{15}\left[\log|t-3| - \log|t|\right] + \text{C} \quad \Rightarrow \quad = \dfrac{1}{15}\left[\log|x^5| - \log|x^5+3|\right] + \text{C}$

So, $\text{I} = \dfrac{1}{3}\log|x| - \dfrac{1}{15}\log|x^5+3| + \text{C}$.

Q19. Evaluate : $\int_0^{2\pi} \dfrac{dx}{1+e^{\sin x}}$.

Sol. Let $\text{I} = \int_0^{2\pi} \dfrac{dx}{1+e^{\sin x}} \quad \ldots\text{(i)} \qquad$ [Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$\Rightarrow \quad = \int_0^{2\pi} \dfrac{dx}{1+e^{\sin(2\pi - x)}} = \int_0^{2\pi} \dfrac{dx}{1+e^{-\sin x}} \qquad \Rightarrow \text{I} = \int_0^{2\pi} \dfrac{e^{\sin x}\, dx}{e^{\sin x}+1} \quad \ldots\text{(ii)}$

Adding (i) and (ii), $2\text{I} = \int_0^{2\pi} \dfrac{1}{1+e^{\sin x}} dx + \int_0^{2\pi} \dfrac{e^{\sin x}}{e^{\sin x}+1} dx$

$\Rightarrow 2\text{I} = \int_0^{2\pi} 1\, dx \qquad \Rightarrow 2\text{I} = [x]_0^{2\pi} = 2\pi - 0 \qquad \Rightarrow \text{I} = \pi$.

Q20. If $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$, then find the value of λ, so that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular vectors.

Sol. We have $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$.

$\therefore \vec{a}+\vec{b} = \hat{i} - \hat{j} + 7\hat{k} + 5\hat{i} - \hat{j} + \lambda\hat{k} = 6\hat{i} - 2\hat{j} + (7+\lambda)\hat{k}$

and $\vec{a}-\vec{b} = \hat{i} - \hat{j} + 7\hat{k} - 5\hat{i} + \hat{j} - \lambda\hat{k} = -4\hat{i} + (7-\lambda)\hat{k}$.

Given $(\vec{a}+\vec{b}) \perp (\vec{a}-\vec{b})$ that implies $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 0$

$\Rightarrow \left[6\hat{i} - 2\hat{j} + (7+\lambda)\hat{k}\right].\left[-4\hat{i} + (7-\lambda)\hat{k}\right] = 0 \quad \Rightarrow -24 + (49 - \lambda^2) = 0$

$\Rightarrow \lambda^2 = 25 \qquad \Rightarrow \lambda = \pm 5$.

Hence, the value of λ is 5 and –5.

Q21. Show that the lines $\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ and $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$ are intersecting. Hence find their point of intersection.

OR Find the vector equation of the plane through the point (2, 1, –1) and (–1, 3, 4) and perpendicular to the plane $x - 2y + 4z = 10$.

Sol. Let the lines are $l : \vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ and $l' : \vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$.

Coordinates of any random point on l are $P(3+\lambda, 2+2\lambda, -4+2\lambda)$ and on l' are $Q(5+3\mu, -2+2\mu, 6\mu)$.

If the lines l and l' intersect then, they must have a common point on them i.e., P and Q must coincide for some values of λ and μ.

Now, $3+\lambda = 5+3\mu$ …(i), $2+2\lambda = -2+2\mu$ …(ii) and $-4+2\lambda = 6\mu \Rightarrow 6\mu - 2\lambda = -4$ …(iii)

Solving (i) and (ii), we get : $\lambda = -4$, $\mu = -2$

Substituting these values in LHS of (iii), $6\mu - 2\lambda = 6(-2) - 2(-4) = -4 = \text{RHS}$

So, the given lines intersect each other.

Now, point of intersection is, P(–1, –6, –12).

OR

Let the d.r.'s of normal vector to the plane be A, B, C.

Then the equation of plane through (2, 1, –1) is $A(x-2) + B(y-1) + C(z+1) = 0$ …(i)

And (–1, 3, 4) lies on plane (i) so, $A(-1-2) + B(3-1) + C(4+1) = 0 \Rightarrow -3A + 2B + 5C = 0$ …(ii)

Also it is given that plane (i) is perpendicular to $x - 2y + 4z = 10$.

So, $A - 2B + 4C = 0$ …(iii) [Using $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Solving (ii) and (iii), $\frac{A}{8+10} = \frac{B}{5+12} = \frac{C}{6-2} \Rightarrow \frac{A}{18} = \frac{B}{17} = \frac{C}{4}$

Substituting the proportionate values of A, B and C in (i), we get :

$18(x-2) + 17(y-1) + 4(z+1) = 0$ i.e., $18x + 17y + 4z = 49$

$\therefore$ vector equation of plane is $\vec{r}.(18\hat{i} + 17\hat{j} + 4\hat{k}) = 49$.

Q22. The probability of two students A and B coming to the school in time are 3/7 and 5/7 respectively. Assuming that the events, 'A coming in time' and 'B coming in time' are independent, find the probability of only one of them coming to the school in time. Write at least one advantage of coming to school in time.

Sol. We have, P(A) = Probability of A coming in time = $\frac{3}{7}$

and, P(B) = Probability of B coming in time = $\frac{5}{7}$.

Probability of only one of them coming to the school in time = $P(A \cap \bar{B}) + P(\bar{A} \cap B)$

$\Rightarrow \quad = P(A)\,P(\bar{B}) + P(\bar{A})\,P(B)$

$\Rightarrow \quad = \frac{3}{7}\left(1 - \frac{5}{7}\right) + \left(1 - \frac{3}{7}\right)\frac{5}{7}$ [$\because P(\bar{E}) = 1 - P(E)$]

$\Rightarrow \quad = \frac{26}{49}$.

Advantage of coming to school in time : Not missing the morning prayers and/or missing any topic in first period of the class.

SECTION – C

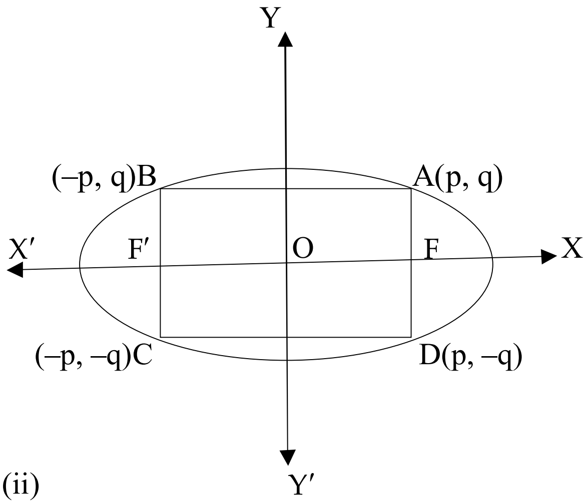
Q23. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

OR Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which pass through the point (4/3, 0).

Sol. Given equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(i)

Let ABCD be the rectangle inscribed in the ellipse.

Let the length of the rectangle be AB = 2p, and width be AD = 2q such that major axis intersects the sides AD and BC at F and F′ respectively.



Since A(p, q) lies on the ellipse (i) so,

$$\frac{p^2}{a^2}+\frac{q^2}{b^2}=1 \Rightarrow q^2=b^2\left(1-\frac{p^2}{a^2}\right) \quad \ldots(ii)$$

Now, area of rectangle A = (2p) (2q)

i.e., $\quad A^2=16p^2q^2$

Assume $A^2=f(p)=16p^2b^2\left(1-\frac{p^2}{a^2}\right)$ [By using (ii)

$\Rightarrow f(p)=\frac{16b^2}{a^2}\left(a^2p^2-p^4\right) \quad \Rightarrow f'(p)=\frac{16b^2}{a^2}\left(2a^2p-4p^3\right)$ and $f''(p)=\frac{16b^2}{a^2}\left(2a^2-12p^2\right)$

For points of local maxima & minima, $f'(p)=0 \Rightarrow \frac{16b^2}{a^2}\left(2a^2p-4p^3\right)=0 \Rightarrow p=0, p=\frac{a}{\sqrt{2}}$

$f''\left(\frac{a}{\sqrt{2}}\right)=\frac{16b^2}{a^2}\left(2a^2-12\times\frac{a^2}{2}\right)=-64b^2<0$ [Value $p=0$ is rejected.

$\therefore f(p)$ is maximum at $p=\frac{a}{\sqrt{2}}$ which means A^2 is maximum at $p=\frac{a}{\sqrt{2}}$.

Hence A is also maximum at $p=\frac{a}{\sqrt{2}}$.

And maximum area is, $A=\sqrt{16\times\frac{a^2}{2}\times b^2\left(1-\frac{1}{2}\right)}=2ab$ Sq.units.

OR Given curve is $3x^2-y^2=8$ …(i). Let the point of contact on (i) be P(x_1, y_1).

So, $3x_1^2-y_1^2=8$ …(A)

Differentiating (i) w.r.t. x, $6x-2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{3x}{y} \qquad \Rightarrow \left.\frac{dy}{dx}\right]_{at\ P}=\frac{3x}{y}$ …(ii)

Equation of tangent at P is : $y-y_1=\frac{3x_1}{y_1}(x-x_1) \qquad \Rightarrow y_1(y-y_1)=3x_1(x-x_1)$ …(iii)

As tangent (iii) passes through (4/3, 0) so,

$y_1(0-y_1)=3x_1\left(\frac{4}{3}-x_1\right) \qquad \Rightarrow -y_1^2=4x_1-3x_1^2 \qquad \Rightarrow 3x_1^2-y_1^2=4x_1$ …(B)

By (A) and (B), $4x_1=8 \qquad \Rightarrow x_1=2,\ y_1=\pm 2$ [By replacing the value of $x_1=2$ in (A)

Hence points of contact are (2, ±2).

$\therefore$ equation of tangents are : $2(y-2)=3(2)(x-2) \Rightarrow 3x-y=4$ [By (iii)

and, $(-2)(y+2)=3(2)(x-2) \Rightarrow 3x+y=4$.

Q24. Find the area of the region bounded by the parabola $y=x^2$ and $y=|x|$.

Sol. Given $y=x^2$ …(i) and, $y=|x|=\begin{cases} x, & \text{if } x\ge 0 \\ -x, & \text{if } x<0 \end{cases}$ …(ii)

Curve (i) is symmetrical about y-axis since it contains even power of x.

Solving (i) and (ii), we get :

Case I : If $x\ge 0$, $x^2=x \Rightarrow x(x-1)=0 \Rightarrow x=0, y=1$

So, points of intersection are (0, 0), (1, 1).

Case II : If $x < 0$, $x^2 = -x \Rightarrow x(x+1) = 0 \qquad \Rightarrow x = 0,\ x = -1$

So, points of intersections are (0, 0), (–1, 1).

Now, required area $= 2 \times ar(\text{OABCO})$

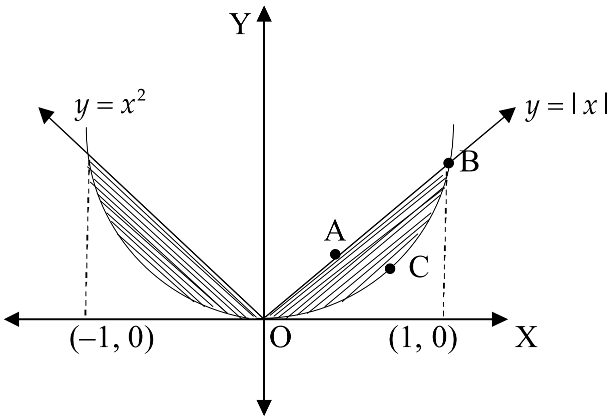
$$\Rightarrow \quad = 2\left[\int_0^1 y_{(ii)}dx - \int_0^1 y_{(i)}dx\right]$$

$$\Rightarrow \quad = 2\left[\int_0^1 |x|\,dx - \int_0^1 x^2 dx\right]$$

$$\Rightarrow \quad = 2\left[\int_0^1 (x - x^2)dx\right]$$

$$\Rightarrow \quad = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$\Rightarrow \quad = 2\left[\left(\frac{1}{2} - \frac{1}{3}\right) - (0 - 0)\right]$$



Required Area $= \dfrac{1}{3}$ Sq.units .

Q25. Find the particular solution of the differential equation $(\tan^{-1} y - x)dy = (1 + y^2)dx$, given that $x = 0$, $y = 0$.

Sol. Given $(\tan^{-1} y - x)dy = (1 + y^2)dx$

$$\Rightarrow \frac{(\tan^{-1} y - x)}{(1 + y^2)} = \frac{dx}{dy} \qquad \Rightarrow \frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{\tan^{-1} y}{1+y^2}$$

It is clear that this is linear differential equation of the form $\dfrac{dx}{dy} + \text{P}(y)x = \text{Q}(y)$

Here $\text{P}(y) = \dfrac{1}{1+y^2}$, $\text{Q}(y) = \dfrac{\tan^{-1} y}{1+y^2}$.

Integrating Factor $= e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}$.

So, solution is given by, $xe^{\tan^{-1} y} = \int e^{\tan^{-1} y}\left(\dfrac{\tan^{-1} y}{1+y^2}\right)dy \qquad$ Put $\tan^{-1} y = t \Rightarrow \dfrac{dy}{1+y^2} = dt$

$$\therefore\ xe^{\tan^{-1} y} = \int e^t t\,dt \qquad \Rightarrow xe^{\tan^{-1} y} = t\int e^t\,dt - \int\left(\frac{d}{dt}(t)\int e^t\,dt\right)dt$$

$$\Rightarrow xe^{\tan^{-1} y} = te^t - e^t + \text{C} \qquad \Rightarrow xe^{\tan^{-1} y} = (\tan^{-1} y - 1)e^{\tan^{-1} y} + \text{C}$$

Given that $x = 0$, $y = 0$, we get : $0.e^{\tan^{-1} 0} = (\tan^{-1} 0 - 1)e^{\tan^{-1} 0} + \text{C} \ \Rightarrow \text{C} = 1$

So, the required solution is : $x = \tan^{-1} y + e^{-\tan^{-1} y} - 1$.

Q26. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r}.(3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from origin is unity.

OR Find the vector equation of the line passing through the point (1, 2, 3) and parallel to the planes $\vec{r}.(\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r}.(3\hat{i} + \hat{j} + \hat{k}) = 6$.

Sol. The equation of the plane passing through the line of intersection of the planes $\vec{r}.(\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r}.(3\hat{i} - \hat{j} - 4\hat{k}) = 0$ is $\vec{r}.[(\hat{i} + 3\hat{j}) + \lambda(3\hat{i} - \hat{j} - 4\hat{k})] = 6 + \lambda(0)$ [Using $\vec{r}.[\vec{n}_1 + \lambda\vec{n}_2] = d_1 + \lambda d_2$

i.e., $\quad \vec{r}.[(1 + 3\lambda)\hat{i} + (3 - \lambda)\hat{j} - 4\lambda\hat{k})] - 6 = 0 \qquad \ldots$(i)

Now distance of (i) from (0, 0) is, $\dfrac{|(0\hat{i} + 0\hat{j} + 0\hat{k}).[(1 + 3\lambda)\hat{i} + (3 - \lambda)\hat{j} - 4\lambda\hat{k})] - 6|}{\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + (-4\lambda)^2}} = 1$

$$\left(\frac{6}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}\right)^2=1 \quad \Rightarrow \frac{36}{26\lambda^2+10}=1 \quad \Rightarrow \lambda=\pm 1.$$

Substituting the value of λ in (i), we get : $\vec{r}.[4\hat{i}+2\hat{j}-4\hat{k}]-6=0$ i.e., $\vec{r}.[2\hat{i}+\hat{j}-2\hat{k})]=3$

and, $\vec{r}.[-2\hat{i}+4\hat{j}+4\hat{k}]-6=0$ i.e., $\vec{r}.[\hat{i}-2\hat{j}-2\hat{k}]+3=0$

OR Required line is parallel to the planes $\vec{r}.(\hat{i}-\hat{j}+2\hat{k})=5$ and $\vec{r}.(3\hat{i}+\hat{j}+\hat{k})=6$.

That implies, the line will be perpendicular to the normals of these planes.

So, line will be parallel to $(\hat{i}-\hat{j}+2\hat{k})\times(3\hat{i}+\hat{j}+\hat{k})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 2\\ 3 & 1 & 1\end{vmatrix}=-3\hat{i}+5\hat{j}+4\hat{k}$.

Now given point on the required line is (1, 2, 3) so, $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$.

So equation of line is : $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(5\hat{j}+4\hat{k}-3\hat{i})$.

Q27. In a hockey match, both teams A and B scored same number of goals up to the end of the game, so to decide the winner, the referee asked both the captains to throw a die alternatively and decided that the team, whose captain gets a six first, will be declared the winner. If the captain of team A was asked to start, find their respective probabilities of winning the match and state whether the decision of the referee was fair or not.

Sol. Let A : team A is declared a winner, B : team B is declared a winner.

Let E : die shows six on the throw.

Clearly P(E) = 1/6, P($\bar{E}$) = 1 – P(E) = 1 – 1/6 = 5/6.

If captain of team A starts then he may get a six in 1st throw or 3rd throw or 5th throw and so on.

$\therefore$ P(A) = P(E) + P($\bar{E}\,\bar{E}$ E) + P($\bar{E}\,\bar{E}\,\bar{E}\,\bar{E}$ E) + …

$$=\frac{1}{6}+\left(\frac{5}{6}\right)^2\frac{1}{6}+\left(\frac{5}{6}\right)^4\frac{1}{6}+\ldots=\frac{\frac{1}{6}}{1-\frac{25}{36}}$$ [Using $S_\infty=\frac{a}{1-r}$, sum of infinite GP

i.e., P(A) = $\frac{6}{11}$

And P(B) = 1 – P(A) = 1 – $\frac{6}{11}=\frac{5}{11}$.

The decision of referee wasn't fair since team A has more chances of being declared a winner despite the fact that both the teams had secured same number of goals.

Q28. A manufacturer considers that men and women workers are equally efficient and so he pays them at the same rate. He has 30 and 17 units of workers (male and female) and capital respectively, which he uses to produce two types of goods A and B. To produce one unit of A, 2 workers and 3 units of capital are required while 3 workers and 1 unit of capital is required to produce one unit of B. If A and B are priced at ₹100 and ₹120 per unit respectively, how should he use his resources to maximize the total revenue? Form the above as an LPP and solve graphically.

Do you agree with this view of the manufacturer that men and women workers are equally efficient and so should be paid at the same rate?

Sol. Let the number of the goods of type A and B produced be respectively x and y.

To maximize, Z = ₹($100x + 120y$)

Subject to the constraints:

$2x+3y\le 30$ …(i) and, $3x+y\le 17$ …(ii); $x, y\ge 0$.

Considering the equations corresponding to the inequations (i) and (ii), we have:

$2x+3y=30$

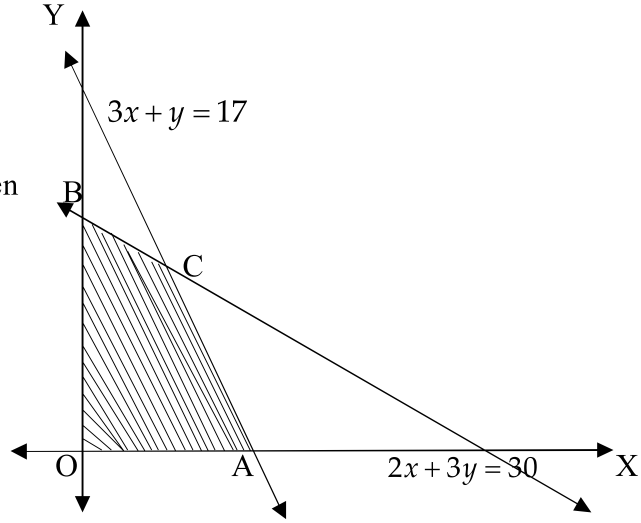
x	15	0
y	0	10

$3x+y=17$

x	17/3	0
y	0	17

Take the testing points as (0, 0) for (i), we have:
$2(0)+3(0)\le 30 \Rightarrow 0\le 30$, which is true.
Take the testing points as (0, 0) for (ii), we have:
$3(0)+(0)\le 17 \Rightarrow 0\le 17$, which is true.

The shaded region **OACBO** as shown in the given figure is the feasible region, which is **bounded**.



The coordinates of the corner points of the feasible region are A(8, 0), B(4, 12), C(0, 14) and O(0, 0).
So, Value of Z at A(17/3, 0) = ₹1700/3
Value of Z at B(0, 10) = ₹1200
Value of Z at C(3, 8) = ₹1260
Value of Z at O(0, 0) = ₹0

The maximum value of Z is ₹1260 which occurs at $x = 3$ and $y = 8$.

Thus, the factory must produce 3 units and 8 units of the goods of type A and B respectively. The maximum obtained profit earned by the factory by producing these items is ₹1260.

Yes, we agree with the view of manufacturer that men and women workers are equally efficient and so should be paid at the same rate.

Q29. The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others, using matrices find the number of awardees of each category. Apart from these values, namely, honesty, cooperation and supervision, suggest one more value which the management of the colony must include for awards.

Sol. Let x, y and z be the number of awardees for honesty, cooperation and supervision respectively.
Acc. to question, $x+y+z=12$, $3(y+z)+2x=33$ i.e. $2x+3y+3z=33$, $x+z=2y$ i.e. $x-2y+z=0$

Clearly $\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 3\\ 1 & -2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}12\\ 33\\ 0\end{bmatrix}$.

By using matrix method: $A=\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 3\\ 1 & -2 & 1\end{bmatrix}$, $B=\begin{bmatrix}12\\ 33\\ 0\end{bmatrix}$ and, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$.

Since $AX=B$ $\Rightarrow A^{-1}(AX)=A^{-1}B$ [Pre-multiplying by A^{-1} both sides
$\therefore (A^{-1}A)X=A^{-1}B$ $\Rightarrow (I)\,X=A^{-1}B$ [$\because IA = A = AI$
$\Rightarrow X=A^{-1}B$ ...(i)

Now, $|A|=\begin{vmatrix}1 & 1 & 1\\ 2 & 3 & 3\\ 1 & -2 & 1\end{vmatrix}=1(3+6)-1(2-3)+1(-4-3)$

$\therefore |A|=3\ne 0$ $\Rightarrow$ A is **non-singular** and hence, it is invertible i.e., A^{-1} exists.

Consider C_{ij} be the cofactors of element a_{ij} in matrix A, we have

$C_{11}=9$, $C_{12}=1$, $C_{13}=-7$
$C_{21}=-3$, $C_{22}=0$, $C_{23}=3$
$C_{31}=0$, $C_{32}=-1$, $C_{33}=1$

So, $adjA = \begin{bmatrix} 9 & 1 & -7 \\ -3 & 0 & 3 \\ 0 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$ $\quad \therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$

Now by (i), we have $X = A^{-1}B$

So, $X = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$ $\Rightarrow = \frac{1}{3}\begin{bmatrix} 108-99+0 \\ 12+0+0 \\ -84+99+0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

By equality of matrices, we have: $x = 3$, $y = 4$, $z = 5$.

$\therefore$ the number of awardees for honesty, cooperation and supervision respectively are 3, 4 and 5.
Another value which the management can include may be **regularity and sincerity**.

CBSE 2013 ALL INDIA EXAMINATION
[Set 2 With Solutions]

Q09. If matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $A^2 = pA$, then write the value of p.

Sol. Given $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $A^2 = pA$

$\therefore \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = p\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ $\Rightarrow \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = p\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = \begin{bmatrix} 2p & -2p \\ -2p & 2p \end{bmatrix}$ [By equality of matrices,

$\therefore p = 4$.

Q10. A and B are two points with position vectors $2\vec{a} - 3\vec{b}$ and $6\vec{b} - \vec{a}$ respectively. Write the position vector of a point P which divides the line segment AB internally in the ratio 1:2.

Sol. We have $\overrightarrow{OA} = 2\vec{a} - 3\vec{b}$ and $\overrightarrow{OB} = 6\vec{b} - \vec{a}$.

Since P divides AB in 1:2 internally so, $\overrightarrow{OP} = \frac{1.\overrightarrow{OB} + 2.\overrightarrow{OA}}{1+2} = \frac{1.(6\vec{b} - \vec{a}) + 2.(2\vec{a} - 3\vec{b})}{1+2}$

i.e, $\overrightarrow{OP} = \frac{3\vec{a}}{3} = \vec{a}$.

Q19. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Sol. Given $x^y = e^{x-y}$.

Taking log on both the sides, $\log(x^y) = \log(e^{x-y})$ $\quad \Rightarrow y\log(x) = (x-y)\log(e) = (x-y)$ $\quad [\therefore \log e = 1$

i.e., $y = \frac{x}{1+\log x}$ $\quad \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{1+\log x}\right)$ [Differentiating w.r.t. x both the sides

$\Rightarrow \frac{dy}{dx} = \frac{(1+\log x)\frac{d}{dx}(x) - x.\frac{d}{dx}(1+\log x)}{(1+\log x)^2}$ $\quad \Rightarrow \frac{dy}{dx} = \frac{(1+\log x)(1) - x.\left(0+\frac{1}{x}\right)}{(1+\log x)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$. **[Hence Proved.**

Q20. Evaluate : $\int \frac{dx}{x(x^3+8)}$.

Sol. Let $I = \int \frac{dx}{x(x^3+8)}$ $\Rightarrow$ $= \frac{1}{3}\int \frac{3x^2dx}{x^3(x^3+8)}$ Put $x^3+8=t \Rightarrow 3x^2dx = dt$

i.e., $I = \frac{1}{3}\int \frac{dt}{(t-8)t}$ $\Rightarrow$ $= \frac{1}{3}\int \frac{1}{8}\times\left(\frac{1}{t-8}-\frac{1}{t}\right)dt$ $\Rightarrow$ $= \frac{1}{24}\left[\log|t-8|-\log|t|\right]+C$

$\therefore I = \frac{1}{24}\log\left|\frac{x^3}{x^3+8}\right|+C$.

Q21. Evaluate : $\int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x}$.

Sol. Let $I = \int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x}$ ...(i) [Using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$\Rightarrow I = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)dx}{1+\cos^2(\pi-x)} = \int_0^{\pi} \frac{(\pi-x)\sin x\,dx}{1+\cos^2 x}$ $\Rightarrow I = \int_0^{\pi} \frac{\pi\sin x dx}{1+\cos^2 x} - \int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x}$ ...(ii)

Adding (i) and (ii), $2I = \pi\int_0^{\pi} \frac{\sin x dx}{1+\cos^2 x}$

Put $\cos x = t \Rightarrow \sin x dx = -dt$. Also when $x=0 \Rightarrow t=\cos 0 = 1$ and $x=\pi \Rightarrow t = \cos\pi = -1$.

$\therefore I = \frac{\pi}{2}\int_1^{-1} \frac{-dt}{1+t^2}$ $\Rightarrow$ $= \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}$ $\Rightarrow$ $= \frac{\pi}{2}\left[\tan^{-1} t\right]_{-1}^{1}$

i.e., $I = \frac{\pi}{2}\left[\tan^{-1}(1)-\tan^{-1}(-1)\right] = \frac{\pi}{2}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]$ $\Rightarrow I = \left(\frac{\pi}{2}\right)^2$.

Q22. If $\vec{p} = 5\hat{i}+\lambda\hat{j}-3\hat{k}$ and $\vec{q} = \hat{i}+3\hat{j}-5\hat{k}$, then find the value of λ, so that $\vec{p}+\vec{q}$ and $\vec{p}-\vec{q}$ are perpendicular vectors.

Sol. We have $\vec{p} = 5\hat{i}+\lambda\hat{j}-3\hat{k}$ and $\vec{q} = \hat{i}+3\hat{j}-5\hat{k}$.

So, $\vec{p}+\vec{q} = 5\hat{i}+\lambda\hat{j}-3\hat{k}+\hat{i}+3\hat{j}-5\hat{k} = 6\hat{i}+(\lambda+3)\hat{j}-8\hat{k}$

And, $\vec{p}-\vec{q} = 5\hat{i}+\lambda\hat{j}-3\hat{k}-\hat{i}-3\hat{j}+5\hat{k} = 4\hat{i}+(\lambda-3)\hat{j}+2\hat{k}$

Since $\vec{p}+\vec{q}$ and $\vec{p}-\vec{q}$ are perpendicular vectors so, $(\vec{p}+\vec{q}).(\vec{p}-\vec{q}) = 0$

$\Rightarrow \left[6\hat{i}+(\lambda+3)\hat{j}-8\hat{k}\right].\left[4\hat{i}+(\lambda-3)\hat{j}+2\hat{k}\right] = 0$ $\Rightarrow 24+(\lambda^2-9)-16 = 0$

$\Rightarrow \lambda^2 = 1$ $\Rightarrow \lambda = \pm 1$.

Q28. Find the area of the region $\{(x,y): y^2 \le 6ax \text{ and } x^2+y^2 \le 16a^2\}$ using method of integration.

Sol. We have $\{(x,y): y^2 \le 6ax \text{ and } x^2+y^2 \le 16a^2\}$.

Consider $y^2 = 6ax$ ...(i), $x^2+y^2 = 16a^2$ ...(ii)

Curve (ii) represents a circle centered at (0, 0) having radius of $4a$.

Solving (i) and (ii), $x^2+6ax-16a^2 = 0$ $\Rightarrow (x+8a)(x-2a) = 0$

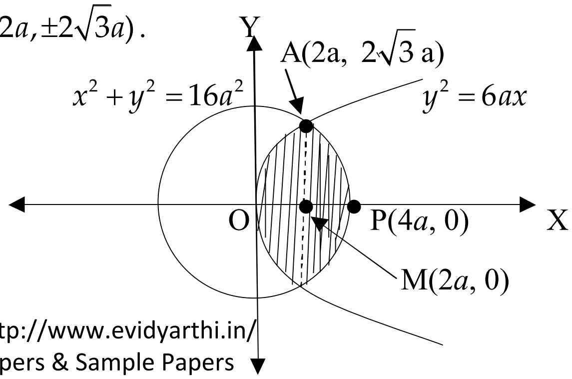
$\therefore x = 2a$ and $x = -8a$ [which is rejected, as it doesn't satisfy (i)]

So, $x = 2a$, $y = \pm 2\sqrt{3}a$ and point of intersections are $(2a, \pm 2\sqrt{3}a)$.

Now required area $= 2\times ar(\text{OAPMO})$

$\Rightarrow$ $= 2\left[\int_0^{2a} y_{(i)}dx + \int_{2a}^{4a} y_{(ii)}dx\right]$

$\Rightarrow$ $= 2\left[\sqrt{6a}\int_0^{2a}\sqrt{x}\,dx + \int_{2a}^{4a}\sqrt{(4a)^2-x^2}\,dx\right]$



$$\Rightarrow \quad = 2\left(\frac{2}{3}\sqrt{6a}\left[x^{3/2}\right]_0^{2a} + \left[\frac{x}{2}\sqrt{(4a)^2 - x^2} + \frac{16a^2}{2}\sin^{-1}\frac{x}{4a}\right]_{2a}^{4a}\right)$$

$$\Rightarrow \quad = \frac{4}{3}\sqrt{6a}\left[(2a)^{3/2} - 0^{3/2}\right] + \left[x\sqrt{(4a)^2 - x^2} + 16a^2\sin^{-1}\frac{x}{4a}\right]_{2a}^{4a}$$

$$\Rightarrow \quad = \frac{4}{3}\sqrt{6a}\sqrt{2a}(2a) + \left[\left(4a\times 0 + 16a^2\sin^{-1}(1)\right) - \left(2a\times 2\sqrt{3}a + 16a^2\sin^{-1}\frac{1}{2}\right)\right]$$

$$\Rightarrow \quad = \frac{4}{3}\sqrt{6a}\sqrt{2a}(2a) + \left[16a^2\left(\frac{\pi}{2}\right) - \left(4\sqrt{3}a^2 + 16a^2\left(\frac{\pi}{6}\right)\right)\right]$$

$$\Rightarrow \quad = \frac{16\sqrt{3}}{3}a^2 + \left[8\pi a^2 - 4\sqrt{3}a^2 - \frac{8\pi}{3}a^2\right]$$

$$\Rightarrow \quad = \frac{4\sqrt{3}}{3}a^2 + \frac{16\pi}{3}a^2 = \frac{4a^2}{3}(4\pi + \sqrt{3})\text{ Sq.units}.$$

Q29. Show that the differential equation $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + xdy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $y = \frac{\pi}{4}$ when $x = 1$.

Sol. We have $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + xdy = 0$ $\quad\Rightarrow \frac{dy}{dx} = -\frac{1}{x}\left[x\sin^2\left(\frac{y}{x}\right) - y\right] = f(x,y)$ (*say*) …(i)

Put $x = kx$ & $y = ky$ in (i), $f(kx,ky) = -\frac{1}{kx}\left[kx\sin^2\left(\frac{ky}{kx}\right) - ky\right] = k^0\left(-\frac{1}{x}\left[x\sin^2\left(\frac{y}{x}\right) - y\right]\right) = k^0 f(x,y)$

So, it is clear that the given diff. eq. is homogenous diff. eq. of degree 0.

Now, let $y = vx$ in (i). So, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore\ v + x\frac{dv}{dx} = -\frac{1}{x}\left[x\sin^2\left(\frac{vx}{x}\right) - vx\right] \quad \Rightarrow v + x\frac{dv}{dx} = -\sin^2 v + v$

$\Rightarrow -\int \text{cosec}^2 v\, dv = \int\frac{dx}{x} \quad \Rightarrow \cot v = \log|x| + C \quad \Rightarrow \cot\left(\frac{y}{x}\right) = \log|x| + C.$

Since given that $y = \frac{\pi}{4}$ when $x = 1$ so, $\cot\frac{\pi}{4} = \log|1| + C \quad \Rightarrow C = 1$

$\therefore$ required solution is $\cot\left(\frac{y}{x}\right) = \log|x| + 1$.

CBSE 2013 ALL INDIA EXAMINATION
[Set 3 With Solutions]

Q09. If matrix $A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ and $A^2 = \lambda A$, then write the value of λ.

Sol. Given $A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ and $A^2 = \lambda A$

$\therefore \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = \lambda\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix} = \lambda\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix} = \begin{bmatrix} 3\lambda & -3\lambda \\ -3\lambda & 3\lambda \end{bmatrix}$ [By equality of matrices,

$\Rightarrow 3\lambda = 18 \quad \therefore \lambda = 6$.

Q10. L and M are two points with position vectors $2\vec{a}-\vec{b}$ and $\vec{a}+2\vec{b}$ respectively. Write the position vector of a point N which divides the line segment LM in the ratio 1:2 externally.

Sol. We have $\overrightarrow{OL}=2\vec{a}-\vec{b}$ and $\overrightarrow{OM}=\vec{a}+2\vec{b}$.

Since N divides LM in 2:1 externally so, $\overrightarrow{ON}=\dfrac{1.\overrightarrow{OM}-2.\overrightarrow{OL}}{1-2}=2.(\vec{a}+2\vec{b})-1.(2\vec{a}-\vec{b})$

i.e, $\overrightarrow{OP}=5\vec{b}$.

Q19. Using vectors, find the area of the triangle ABC, whose vertices are A(1,2,3), B(2,–1,4) and C(4,5,–1).

Sol. Given vertices of Δ ABC are A(1,2,3), B(2,–1,4) and C(4,5,–1).

So, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(2\hat{i}-\hat{j}+4\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})=\hat{i}-3\hat{j}+\hat{k}$ and,

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(4\hat{i}+5\hat{j}-\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})=3\hat{i}+3\hat{j}-4\hat{k}$

Now $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{vmatrix}=9\hat{i}+7\hat{j}+12\hat{k}$ $\Rightarrow|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{9^2+7^2+12^2}=\sqrt{274}$

$ar(\text{ABC})=\dfrac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\dfrac{1}{2}\sqrt{274}$ Sq.units.

Q20. Evaluate : $\displaystyle\int\frac{dx}{x(x^3+1)}$.

Sol. Let $\mathrm{I}=\displaystyle\int\frac{dx}{x(x^3+1)}$ $\Rightarrow$ $=\displaystyle\frac{1}{3}\int\frac{3x^2dx}{x^3(x^3+1)}$ Put $x^3+1=t\Rightarrow 3x^2dx=dt$

i.e., $\mathrm{I}=\displaystyle\frac{1}{3}\int\frac{dt}{(t-1)t}$ $\Rightarrow$ $=\displaystyle\frac{1}{3}\int\left(\frac{1}{t-1}-\frac{1}{t}\right)dt$ $\Rightarrow$ $=\dfrac{1}{3}\left[\log|t-1|-\log|t|\right]+\mathrm{C}$

$\therefore \mathrm{I}=\dfrac{1}{3}\log\left|\dfrac{x^3}{x^3+1}\right|+\mathrm{C}$.

Q21. If $x\sin(a+y)+\sin a\cos(a+y)=0$, prove that $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$.

Sol. Given $x\sin(a+y)+\sin a\cos(a+y)=0$ $\Rightarrow x=-\sin a\dfrac{\cos(a+y)}{\sin(a+y)}$

i.e., $\dfrac{dx}{dy}=-\sin a\times\dfrac{d}{dy}\left(\dfrac{\cos(a+y)}{\sin(a+y)}\right)$ [Differentiating w.r.t. y both the sides

$$\Rightarrow\frac{dx}{dy}=-\sin a\times\frac{\sin(a+y)\frac{d}{dy}\left(\cos(a+y)\right)-\cos(a+y)\frac{d}{dy}\left(\sin(a+y)\right)}{[\sin(a+y)]^2}$$

$$\Rightarrow\frac{dx}{dy}=-\sin a\times\frac{-\sin(a+y)\sin(a+y)-\cos(a+y)\cos(a+y)}{[\sin(a+y)]^2}=\sin a\times\left[\frac{\sin^2(a+y)+\cos^2(a+y)}{[\sin(a+y)]^2}\right]$$

$\Rightarrow\dfrac{dx}{dy}=\sin a\times\dfrac{1}{\sin^2(a+y)}$ $\therefore\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$. **[Hence Proved.**

Q22. Using properties of determinants, prove that $\begin{vmatrix}3x & -x+y & -x+z\\ x-y & 3y & z-y\\ x-z & y-z & 3z\end{vmatrix}=3(x+y+z)(xy+yz+zx)$.

Sol. LHS: Let $\Delta=\begin{vmatrix}3x & -x+y & -x+z\\ x-y & 3y & z-y\\ x-z & y-z & 3z\end{vmatrix}$ [Applying $C_1\to C_1+C_2+C_3$

$$\Rightarrow \quad = \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$ [Taking $x+y+z$ common from C_1

$$\Rightarrow \quad = (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$ [Applying $R_1 \to R_1 - R_3, R_2 \to R_2 - R_3$

$$\Rightarrow \quad = (x+y+z)\begin{vmatrix} 0 & -x+z & -x-2z \\ 0 & 2y+z & -2z-y \\ 1 & y-z & 3z \end{vmatrix}$$ [Expanding along C_1

$$\Rightarrow \quad = (x+y+z)\left(1 \times \begin{vmatrix} -x+z & -x-2z \\ 2y+z & -2z-y \end{vmatrix}\right)$$

$$\Rightarrow \quad = (x+y+z)\big((-x+z)(-2z-y)-(2y+z)(-x-2z)\big)$$

$$\Rightarrow \quad = (x+y+z)(3xy+3yz+3zx)$$

$$\Rightarrow \quad = 3(x+y+z)(xy+yz+zx) = \text{RHS.}$$ **[Hence Proved.**

Q28. Find the area of the region $\{(x,y): y^2 \le 4x \text{ and } 4x^2+4y^2 \le 9\}$ using method of integration.

Sol. We have $\{(x,y): y^2 \le 4x \text{ and } 4x^2+4y^2 \le 9\}$.

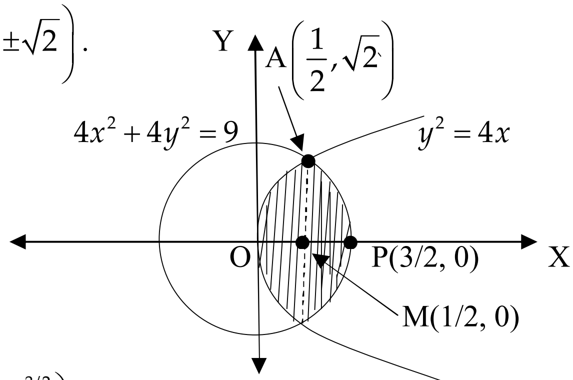
Consider $y^2 = 4x$ ...(i), $4x^2+4y^2 = 9 \Rightarrow x^2+y^2 = \left(\frac{3}{2}\right)^2$ ...(ii)

Curve (ii) represents a circle centered at (0, 0) having radius of 3/2 units.

Solving (i) and (ii), $4x^2+4(4x)=9 \quad \Rightarrow 4x^2+16x-9=0 \quad \Rightarrow (2x+9)(2x-1)=0$

$\therefore x = \frac{1}{2}$ and $x = -\frac{9}{2}$ [which is rejected, as it doesn't satisfy (i)]

So, $x = \frac{1}{2}$, $y = \pm\sqrt{2}$ and point of intersections are $\left(\frac{1}{2}, \pm\sqrt{2}\right)$.



Now required area $= 2 \times ar(\text{OAPMO})$

$$\Rightarrow \quad = 2\left[\int_0^{1/2} y_{(i)}dx + \int_{1/2}^{3/2} y_{(ii)}dx\right]$$

$$\Rightarrow \quad = 2\left[2\int_0^{1/2}\sqrt{x}dx + \int_{1/2}^{3/2}\sqrt{\left(\frac{3}{2}\right)^2 - x^2}\,dx\right]$$

$$\Rightarrow \quad = 2\left(\frac{2}{3}\times 2\left[x^{3/2}\right]_0^{1/2} + \left[\frac{x}{2}\sqrt{\left(\frac{3}{2}\right)^2 - x^2} + \frac{9}{8}\sin^{-1}\frac{2x}{3}\right]_{1/2}^{3/2}\right)$$

$$\Rightarrow \quad = \frac{8}{3}\left[\frac{1}{2\sqrt{2}} - 0^{3/2}\right] + \left[x\sqrt{\frac{9}{4}-x^2} + \frac{9}{4}\sin^{-1}\frac{2x}{3}\right]_{1/2}^{3/2}$$

$$\Rightarrow \quad = \frac{4}{3\sqrt{2}} + \left[\left(\frac{3}{2}\times 0 + \frac{9}{4}\sin^{-1}(1)\right) - \left(\frac{1}{2}\times\sqrt{2} + \frac{9}{4}\sin^{-1}\frac{1}{3}\right)\right]$$

$$\Rightarrow \quad = \frac{4}{3\sqrt{2}} + \frac{9}{4}\times\frac{\pi}{2} - \frac{1}{\sqrt{2}} - \frac{9}{4}\sin^{-1}\frac{1}{3}$$

$$\Rightarrow \quad = \frac{1}{3\sqrt{2}} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}$$

$$\Rightarrow \quad = \frac{1}{3\sqrt{2}} + \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right) = \left(\frac{1}{3\sqrt{2}} + \frac{9}{4}\cos^{-1}\frac{1}{3}\right) \text{Sq.units.}$$

Q29. Find the particular solution of the differential equation $\frac{dx}{dy}+x\cot y=2y+y^2\cot y,\ (y\neq 0)$, given that $x=0$ when $y=\frac{\pi}{2}$.

Sol. We have $\frac{dx}{dy}+x\cot y=2y+y^2\cot y,\ (y\neq 0)$

It is clear that this is linear differential equation of the form $\frac{dx}{dy}+\mathrm{P}(y)x=\mathrm{Q}(y)$

Here $\mathrm{P}(y)=\cot y$, $\mathrm{Q}(y)=2y+y^2\cot y$.

Integrating Factor $=e^{\int \cot y dy}=e^{\log\sin y}=\sin y$.

So, solution is given by, $x\sin y=\int \sin y\left(2y+y^2\cot y\right)dy$

$\Rightarrow x\sin y=\int 2y\sin y dy+\int y^2\cos y dy$ [Applying By parts in 2nd integral

$\Rightarrow x\sin y=\int 2y\sin y dy+y^2\int \cos y dy-\int\left(\frac{d}{dy}(y^2)\int \cos y dy\right)dy$

$\Rightarrow x\sin y=\int 2y\sin y dy+y^2\sin y-\int 2y\sin y dy$ $\Rightarrow x\sin y=y^2\sin y+\mathrm{C}$

Given that $x=0$ when $y=\frac{\pi}{2}$, we get : $0\sin\frac{\pi}{2}=\left(\frac{\pi}{2}\right)^2\sin\frac{\pi}{2}+\mathrm{C}$ $\Rightarrow \mathrm{C}=\frac{\pi^2}{4}$

So, the required solution is : $x\sin y=y^2\sin y+\frac{\pi^2}{4}$ i.e., $4(x-y^2)\sin y=\pi^2$