Solutions to IITJEE–2006

Mathematics

Time: 2 hours

Note: Question number 1 to 12 carries (3, -1) ***marks*** each, 13 to 20 carries (5, -1) ***marks*** each, 21 to 32 carries (5, -2) ***marks*** each and 33 to 40 carries (6, 0) ***marks*** each.

Section – A (Single Option Correct)

1. For x > 0, $\lim\limits_{x\to 0}\left((\sin x)^{1/x} + (1/x)^{\sin x}\right)$ is

(A) 0 (B) – 1

(C) 1 (D) 2

Sol. **(C)**

$$\lim_{x\to 0}\left((\sin x)^{1/x} + \left(\frac{1}{x}\right)^{\sin x}\right)$$

$0 + e^{\lim\limits_{x\to 0}\sin x \ln\left(\frac{1}{x}\right)} = 1$ (using L' Hospital's rule).

2. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx$ is equal to

(A) $\frac{\sqrt{2x^4-2x^2+1}}{x^2} + c$ (B) $\frac{\sqrt{2x^4-2x^2+1}}{x^3} + c$

(C) $\frac{\sqrt{2x^4-2x^2+1}}{x} + c$ (D) $\frac{\sqrt{2x^4-2x^2+1}}{2x^2} + c$

Sol. **(D)**

$$\int \frac{\left(\frac{1}{x^3} - \frac{1}{x^5}\right)dx}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}$$

Let $2 - \frac{2}{x^2} + \frac{1}{x^4} = z \Rightarrow \frac{1}{4}\int \frac{dz}{\sqrt{z}}$

$\Rightarrow \frac{1}{2} \times \sqrt{z} + c \Rightarrow \frac{1}{2}\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}} + c$.

3. Given an isosceles triangle, whose one angle is 120° and radius of its incircle = $\sqrt{3}$. Then the area of the triangle in sq. units is

(A) $7 + 12\sqrt{3}$ (B) $12 - 7\sqrt{3}$

(C) $12 + 7\sqrt{3}$ (D) 4π

Sol. **(C)**

$\Delta = \frac{\sqrt{3}}{4}b^2$ …(1)

Also $\frac{\sin 120°}{a} = \frac{\sin 30°}{b} \Rightarrow a = \sqrt{3}b$

and $\Delta = \sqrt{3}s$ and $s = \frac{1}{2}(a+2b)$

$\Rightarrow \Delta = \frac{\sqrt{3}}{2}(a+2b)$ ...(2)

From (1) and (2), we get $\Delta = \left(12+7\sqrt{3}\right)$.

4. If $0 < \theta < 2\pi$, then the intervals of values of θ for which $2\sin^2\theta - 5\sin\theta + 2 > 0$, is

(A) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ (B) $\left(\frac{\pi}{8}, \frac{5\pi}{6}\right)$

(C) $\left(0, \frac{\pi}{8}\right) \cup \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (D) $\left(\frac{41\pi}{48}, \pi\right)$

Sol. **(A)**

$2\sin^2\theta - 5\sin\theta + 2 > 0$

$\Rightarrow (\sin\theta - 2)(2\sin\theta - 1) > 0$

$\Rightarrow \sin\theta < \frac{1}{2}$

$\Rightarrow \theta \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$.

5. If $w = \alpha + i\beta$, where $\beta \neq 0$ and $z \neq 1$, satisfies the condition that $\left(\frac{w - \bar{w}z}{1-z}\right)$ is purely real, then the set of values of z is

(A) $\{z : |z| = 1\}$ (B) $\{z : z = \bar{z}\}$

(C) $\{z : z \neq 1\}$ (D) $\{z : |z| = 1, z \neq 1\}$

Sol. **(D)**

$\frac{w - \bar{w}z}{1-z} = \frac{\bar{w} - w\bar{z}}{1-\bar{z}}$

$\Rightarrow (z\bar{z} - 1)(\bar{w} - w) = 0$

$\Rightarrow z\bar{z} = 1 \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1$.

6. Let a, b, c be the sides of a triangle. No two of them are equal and $\lambda \in R$. If the roots of the equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real, then

(A) $\lambda < \frac{4}{3}$ (B) $\lambda > \frac{5}{3}$

(C) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (D) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

Sol. **(A)**

$D \geq 0$

$\Rightarrow 4(a + b + c)^2 - 12\lambda(ab + bc + ca) \geq 0$

$\Rightarrow \lambda \leq \frac{a^2+b^2+c^2}{3(ab+bc+ca)} + \frac{2}{3}$

Since $|a - b| < c \Rightarrow a^2 + b^2 - 2ab < c^2$ ...(1)

$|b - c| < a \Rightarrow b^2 + c^2 - 2bc < a^2$ ...(2)

$|c - a| < b \Rightarrow c^2 + a^2 - 2ac < b^2$ ...(3)

From (1), (2) and (3), we get $\frac{a^2+b^2+c^2}{ab+bc+ca} < 2$.

Hence $\lambda < \frac{2}{3} + \frac{2}{3} \Rightarrow \lambda < \frac{4}{3}$.

7. If $f''(x) = -f(x)$ and $g(x) = f'(x)$ and $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ and given that $F(5) = 5$, then $F(10)$ is equal to

(A) 5 (B) 10
(C) 0 (D) 15

Sol. **(A)**
$f''(x) = -f(x)$ and $f'(x) = g(x)$
$\Rightarrow f''(x) . f'(x) + f(x) . f'(x) = 0$
$\Rightarrow f(x)^2 + (f'(x))^2 = c \Rightarrow (f(x))^2 + (g(x))^2 = c$
$\Rightarrow F(x) = c \Rightarrow F(10) = 5.$

8. If r, s, t are prime numbers and p, q are the positive integers such that the LCM of p, q is $r^2t^4s^2$, then the number of ordered pair (p, q) is
(A) 252 (B) 254
(C) 225 (D) 224

Sol. **(C)**
Required number of ordered pair (p, q) is $(2 \times 3 - 1)(2 \times 5 - 1)(2 \times 3 - 1) = 225$.

9. Let $\theta \in \left(0, \frac{\pi}{4}\right)$ and $t_1 = (\tan\theta)^{\tan\theta}$, $t_2 = (\tan\theta)^{\cot\theta}$, $t_3 = (\cot\theta)^{\tan\theta}$ and $t_4 = (\cot\theta)^{\cot\theta}$, then

(A) $t_1 > t_2 > t_3 > t_4$ (B) $t_4 > t_3 > t_1 > t_2$
(C) $t_3 > t_1 > t_2 > t_4$ (D) $t_2 > t_3 > t_1 > t_4$

Sol. **(B)**

Given $\theta \in \left(0, \frac{\pi}{4}\right)$, then $\tan\theta < 1$ and $\cot\theta > 1$.

Let $\tan\theta = 1 - \lambda_1$ and $\cot\theta = 1 + \lambda_2$ where λ_1 and λ_2 are very small and positive.

then $t_1 = (1-\lambda_1)^{1-\lambda_1}$, $t_2 = (1-\lambda_1)^{1+\lambda_2}$

$t_3 = (1+\lambda_2)^{1-\lambda_1}$ and $t_4 = (1+\lambda_2)^{1+\lambda_2}$

Hence $t_4 > t_3 > t_1 > t_2$.

10. The axis of a parabola is along the line y = x and the distance of its vertex from origin is $\sqrt{2}$ and that from its focus is $2\sqrt{2}$. If vertex and focus both lie in the first quadrant, then the equation of the parabola is
(A) $(x + y)^2 = (x - y - 2)$ (B) $(x - y)^2 = (x + y - 2)$
(C) $(x - y)^2 = 4(x + y - 2)$ (D) $(x - y)^2 = 8(x + y - 2)$

Sol. **(D)**
Equation of directrix is $x + y = 0$.
Hence equation of the parabola is

$$\frac{x+y}{\sqrt{2}} = \sqrt{(x-2)^2 + (y-2)^2}$$

Hence equation of parabola is
$(x - y)^2 = 8(x + y - 2)$.

11. A plane passes through (1, – 2, 1) and is perpendicular to two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$. The distance of the plane from the point (1, 2, 2) is
(A) 0 (B) 1
(C) $\sqrt{2}$ (D) $2\sqrt{2}$

Sol. **(D)**
The plane is $a(x - 1) + b(y + 2) + c(z - 1) = 0$
where $2a - 2b + c = 0$ and $a - b + 2c = 0$

$\Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{0}$

So, the equation of plane is $x + y + 1 = 0$

$\therefore$ Distance of the plane from the point (1, 2, 2) = $\frac{1+2+1}{\sqrt{1^2+1^2}} = 2\sqrt{2}$.

12. Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$. A vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is

(A) $4\hat{i} - \hat{j} + 4\hat{k}$ (B) $3\hat{i} + \hat{j} - 3\hat{k}$
(C) $2\hat{i} + \hat{j} - 2\hat{k}$ (D) $4\hat{i} + \hat{j} - 4\hat{k}$

Sol. **(A)**

Vector lying in the plane of $\vec{a}$ and $\vec{b}$ is $\vec{r} = \lambda_1\vec{a} + \lambda_2\vec{b}$ and its projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$

$\Rightarrow \left[(\lambda_1+\lambda_2)\hat{i} - (2\lambda_1 - \lambda_2)\hat{j} + (\lambda_1+\lambda_2)\hat{k}\right] \cdot \frac{\left[\hat{i} - \hat{j} - \hat{k}\right]}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\Rightarrow 2\lambda_1 - \lambda_2 = -1 \Rightarrow \vec{r} = (3\lambda_1 + 1)\hat{i} - \hat{j} + (3\lambda_1 + 1)\hat{k}$

Hence the required vector is $4\hat{i} - \hat{j} + 4\hat{k}$.

Alternate:

Vector lying in the plane of $\vec{a}$ and $\vec{b}$ is $\vec{a} + \lambda\vec{b}$, and its projection on C is $\frac{1}{\sqrt{3}}$.

$\Rightarrow \left((1+\lambda)\hat{i} + (2-\lambda)\hat{j} + (1+\lambda)\hat{k} \cdot \frac{(\hat{i} - \hat{j} - \hat{k})}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}$

$\Rightarrow \lambda = 3$.

Hence the required vector is $4\hat{i} - \hat{j} + 4\hat{k}$.

Section – B (May have more than one option correct)

13. The equations of the common tangents to the parabola $y = x^2$ and $y = -(x-2)^2$ is/are

(A) $y = 4(x-1)$ (B) $y = 0$
(C) $y = -4(x-1)$ (D) $y = -30x - 50$

Sol. **(A), (B)**

Equation of tangent to $x^2 = y$ is

$y = mx - \frac{1}{4}m^2$ …(1)

Equation of tangent to $(x-2)^2 = -y$ is

$y = m(x-2) + \frac{1}{4}m^2$ …(2)

(1) and (2) are identical.

$\Rightarrow m = 0$ or 4

$\therefore$ Common tangents are $y = 0$ and $y = 4x - 4$.

14. If $f(x) = \min\{1, x^2, x^3\}$, then

(A) f(x) is continuous $\forall\, x \in R$ (B) $f'(x) > 0,\ \forall\, x > 1$
(C) f(x) is not differentiable but continuous $\forall\, x \in R$ (D) f(x) is not differentiable for two values of x

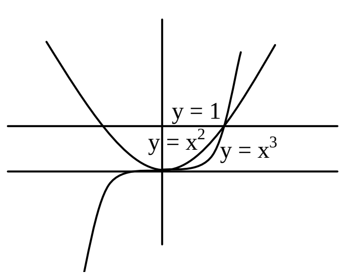
Sol. **(A), (C)**

$f(x) = \min.\{1, x^2, x^3\}$

$\Rightarrow f(x) = \begin{cases} x^3, & x \le 1 \\ 1, & x > 1 \end{cases}$

$\Rightarrow$ f(x) is continuous $\forall\, x \in R$

and non-differentiable at x = 1.



15. A tangent drawn to the curve y = f(x) at P(x, y) cuts the x-axis and y-axis at A and B respectively such that BP : AP = 3 : 1, given that f(1) = 1, then

(A) equation of curve is $x\frac{dy}{dx} - 3y = 0$

(B) normal at (1, 1) is x + 3y = 4

(C) curve passes through (2, 1/8)

(D) equation of curve is $x\frac{dy}{dx} + 3y = 0$

Sol. **(C), (D)**

Equation of the tangent is

$$Y - y = \frac{dy}{dx}(X - x)$$

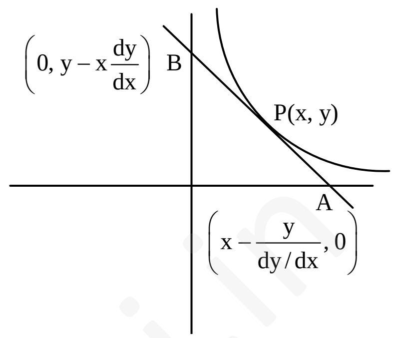
Given $\frac{BP}{AP} = \frac{3}{1}$ so that

$\Rightarrow \frac{dx}{x} = -\frac{dy}{3y} \Rightarrow x\frac{dy}{dx} + 3y = 0$

$\Rightarrow \ln x = -\frac{1}{3}\ln y - \ln c \Rightarrow \ln x^3 = -(\ln cy)$

$\Rightarrow \frac{1}{x^3} = cy$. Given f(1) = 1 $\Rightarrow$ c = 1

$\therefore y = \frac{1}{x^3}$.



16. If a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and its transverse and conjugate axes coincide with the major and minor axes of the ellipse, and the product of eccentricities is 1, then

(A) the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

(B) the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$

(C) focus of hyperbola is (5, 0)

(D) focus of hyperbola is $(5\sqrt{3}, 0)$

Sol. **(A), (C)**

Eccentricity of ellipse = $\frac{3}{5}$

Eccentricity of hyperbola = $\frac{5}{3}$ and it passes through (± 3, 0)

$\Rightarrow$ its equation $\frac{x^2}{9} - \frac{y^2}{b^2} = 1$

where $1 + \frac{b^2}{9} = \frac{25}{9} \Rightarrow b^2 = 16$

$\Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$ and its foci are (±5, 0).

17. Internal bisector of ∠A of triangle ABC meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at E and the side AB at F. If a, b, c represent sides of ΔABC then

(A) AE is HM of b and c

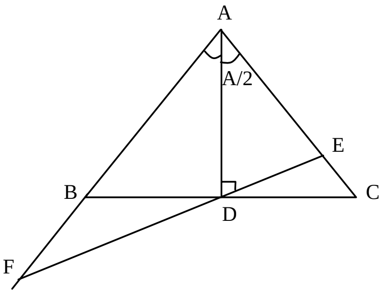
(B) $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

(C) $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$

(D) the triangle AEF is isosceles

Sol. **(A), (B), (C), (D).**

We have ΔABC = ΔABD + ΔACD



$\Rightarrow \quad \frac{1}{2}bc\sin A = \frac{1}{2}cAD\sin\frac{A}{2} + \frac{1}{2}b \times AD\sin\frac{A}{2}$

$\Rightarrow \quad AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

Again AE = AD sec $\frac{A}{2}$

$= \frac{2bc}{b+c} \quad \Rightarrow$ AE is HM of b and c.

EF = ED + DF = 2DE = 2 × AD tan $\frac{A}{2}$ = $\frac{2\times 2bc}{b+c}\times\cos\frac{A}{2}\times\tan\frac{A}{2}$

$= \frac{4bc}{b+c}\sin\frac{A}{2}$

As AD ⊥ EF and DE = DF and AD is bisector ⇒ AEF is isosceles.
Hence A, B, C and D are correct answers.

18. f(x) is cubic polynomial which has local maximum at x = – 1. If f(2) = 18, f(1) = –1 and f′(x) has local minima at x = 0, then

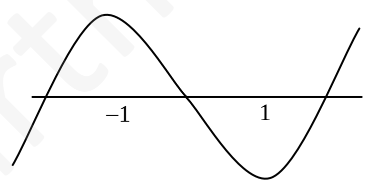
(A) the distance between (–1, 2) and (a, f(a)), where x = a is the point of local minima is $2\sqrt{5}$
(B) f(x) is increasing for x ∈ [1, $2\sqrt{5}$]
(C) f(x) has local minima at x = 1
(D) the value of f(0) =5

Sol. **(B), (C)**

The required polynomial which satisfy the condition is f(x) = $\frac{1}{4}$ $(19x^3 – 57x + 34)$

f(x) has local maximum at x = –1 and local minimum at x = 1

Hence f(x) is increasing for x ∈ $\left[1, 2\sqrt{5}\right]$.



19. Let $\vec{A}$ be vector parallel to line of intersection of planes P_1 and P_2 through origin. P_1 is parallel to the vectors $2\hat{j}+3\hat{k}$ and $4\hat{j}-3\hat{k}$ and P_2 is parallel to $\hat{j}-\hat{k}$ and $3\hat{i}+3\hat{j}$, then the angle between vectors $\vec{A}$ and $2\hat{i}+\hat{j}-2\hat{k}$ is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{6}$ (D) $\frac{3\pi}{4}$

Sol. **(B), (D)**

Vector AB is parallel to $\left[(2\hat{i}+3\hat{k})\times(4)-3\hat{k}\right]\times\left[(\hat{j}-\hat{k})\times(3\hat{i}+3\hat{j})\right] = 54(\hat{j}-\hat{k})$

Let θ is the angle between the vector, then

$$\cos\theta = \pm\left(\frac{54+108}{3.54\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2}}$$

Hence $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$.

20. $f(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ 2 - e^{x-1}, & 1 < x \le 2 \\ x - e, & 2 < x \le 3 \end{cases}$ and $g(x) = \int_0^x f(t)\,dt$, x ∈ [1, 3] then g (x) has

(A) local maxima at x = 1 + ln 2 and local minima at x = e
(B) local maxima at x = 1 and local minima at x = 2
(C) no local maxima
(D) no local minima

Sol. **(A), (B)**

$$g'(x) = f(x) = \begin{cases} e^x & 0 \le x \le 1 \\ 2 - e^{x-1} & 1 < x \le 2 \\ x - e & 2 < x \le 3 \end{cases}$$

$g'(x) = 0$, when $x = 1 + \ln 2$ and $x = e$

$$g''(x) = \begin{Bmatrix} -e^{x-1} & 1 < x \le 2 \\ 1 & 2 < x \le 3 \end{Bmatrix}$$

$g''(1 + \ln 2) = -e^{\ln 2} < 0$ hence at $x = 1 + \ln 2$, $g(x)$ has a local maximum

$g''(e) = 1 > 0$ hence at $x = e$, $g(x)$ has local minimum.

$\because$ $f(x)$ is discontinuous at $x = 1$, then we get local maxima at $x = 1$ and local minima at $x = 2$.

Section – C

Comprehension I

There are n urns each containing n + 1 balls such that the ith urn contains i white balls and (n + 1 – i) red balls. Let u_i be the event of selecting ith urn, i = 1, 2, 3 …, n and w denotes the event of getting a white ball.

21. If $P(u_i) \propto i$, where i = 1, 2, 3, …n, then $\lim_{n\to\infty} P(w)$ is equal to

(A) 1
(B) $\frac{2}{3}$
(C) $\frac{3}{4}$
(D) $\frac{1}{4}$

Sol. **(B)**

$P(u_i) = ki$

$\Sigma P(u_i) = 1$

$\Rightarrow \quad k = \frac{2}{n(n+1)}$

$$\lim_{n\to\infty} P(w) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2i^2}{n(n+1)^2} = \lim_{n\to\infty} \frac{2n(n+1)(2n+1)}{n(n+1)^2 6} = \frac{2}{3}$$

22. If $P(u_i) = c$, where c is a constant then $P(u_n/w)$ is equal to

(A) $\frac{2}{n+1}$
(B) $\frac{1}{n+1}$
(C) $\frac{n}{n+1}$
(D) $\frac{1}{2}$

Sol. **(A)**

$$P\left(\frac{u_n}{w}\right) = \frac{c\left(\frac{n}{n+1}\right)}{c\left(\frac{\Sigma i}{(n+1)}\right)} = \frac{2}{n+1}.$$

23. If n is even and E denotes the event of choosing even numbered urn ($P(u_i) = \frac{1}{n}$), then the value of $P(w/E)$ is

(A) $\frac{n+2}{2n+1}$
(B) $\frac{n+2}{2(n+1)}$
(C) $\frac{n}{n+1}$
(D) $\frac{1}{n+1}$

Sol. **(B)**

$$P\left(\frac{W}{E}\right) = \frac{2+4+6+\cdots n}{\frac{n(n+1)}{2}} = \frac{n+2}{2(n+1)}$$

Comprehension II

Suppose we define the definite integral using the following formula $\int_a^b f(x)dx = \frac{b-a}{2}(f(a)+f(b))$, for more accurate result for $c \in (a, b)$ $F(c) = \frac{c-a}{2}(f(a)+f(c)) + \frac{b-c}{2}(f(b)+f(c))$. When $c = \frac{a+b}{2}$, $\int_a^b f(x)dx = \frac{b-a}{4}(f(a)+f(b)+2f(c))$.

24. $\int_0^{\pi/2} \sin x\, dx$ is equal to

(A) $\frac{\pi}{8}(1+\sqrt{2})$ (B) $\frac{\pi}{4}(1+\sqrt{2})$

(C) $\frac{\pi}{8\sqrt{2}}$ (D) $\frac{\pi}{4\sqrt{2}}$

Sol. **(A)**

$$\int_0^{\pi/2} \sin x\, dx = \frac{\frac{\pi}{2}+0}{4}\left(\sin(0) + \sin\left(\frac{\pi}{2}\right) + 2\sin\left(\frac{0+\frac{\pi}{2}}{2}\right)\right)$$

$$= \frac{\pi}{8}(1+\sqrt{2}).$$

25. Data could not be retrieved.

26. If $f''(x) < 0\ \forall\ x \in (a, b)$ and c is a point such that $a < c < b$, and $(c, f(c))$ is the point lying on the curve for which F(c) is maximum, then $f'(c)$ is equal to

(A) $\frac{f(b)-f(a)}{b-a}$ (B) $\frac{2(f(b)-f(a))}{b-a}$

(C) $\frac{2f(b)-f(a)}{2b-a}$ (D) 0

Sol. **(A)**

$(F'(c) = (b-a)f'(c) + f(a) - f(b)$

$F''(c) = f''(c)(b-a) < 0$

$\Rightarrow\ F'(c) = 0\ \Rightarrow\ f'(c) = \frac{f(b)-f(a)}{b-a}$.

Comprehension III

Let ABCD be a square of side length 2 units. C_2 is the circle through vertices A, B, C, D and C_1 is the circle touching all the sides of the square ABCD. L is a line through A.

27. If P is a point on C_1 and Q in another point on C_2, then $\frac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2}$ is equal to

(A) 0.75 (B) 1.25

(C) 1 (D) 0.5

Sol. **(A)**

Let A, B, C and D be the complex numbers $\sqrt{2}$, $-\sqrt{2}$, $\sqrt{2}i$ and $-\sqrt{2}i$ respectively.

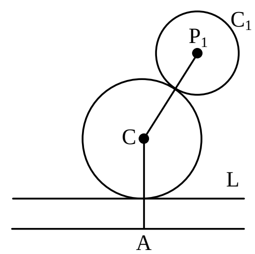
$$\Rightarrow \frac{PA^2+PB^2+PC^2+PD^2}{QA^2+QB^2+QC^2+QD^2} = \frac{|z_1-\sqrt{2}|^2+|z_1+\sqrt{2}|^2+|z_1+\sqrt{2}i|^2+|z_1-\sqrt{2}i|^2}{|z_2+\sqrt{2}|^2+|z_2-\sqrt{2}|^2+|z_2-\sqrt{2}i|^2+|z_2+\sqrt{2}i|^2} = \frac{|z_1|^2+2}{|z_2|^2+2} = \frac{3}{4}.$$

28. A circle touches the line L and the circle C_1 externally such that both the circles are on the same side of the line, then the locus of centre of the circle is

(A) ellipse
(B) hyperbola
(C) parabola
(D) parts of straight line

Sol. **(C)**

Let C be the centre of the required circle.
Now draw a line parallel to L at a distance of r_1 (radius of C_1) from it.
Now $CP_1 = AC \Rightarrow$ C lies on a parabola.



29. A line M through A is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex A are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1, then area of $\Delta T_1T_2T_3$ is

(A) $\frac{1}{2}$ sq. units
(B) $\frac{2}{3}$ sq. units
(C) 1 sq. unit
(D) 2 sq. units

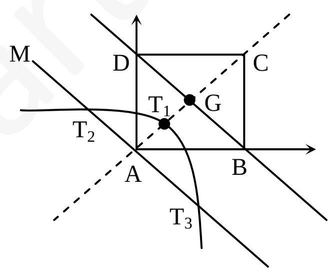
Sol. **(C)**

$\because$ AG = $\sqrt{2}$

$\therefore AT_1 = T_1G = \frac{1}{\sqrt{2}}$ [as A is the focus, T_1 is the vertex and BD is the directrix of parabola].

Also T_2T_3 is latus rectum $\therefore T_2T_3 = 4 \times \frac{1}{\sqrt{2}}$

$\therefore$ Area of $\Delta T_1T_2T_3 = \frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{4}{\sqrt{2}} = 1$.



Comprehension IV

$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, if U_1, U_2 and U_3 are columns matrices satisfying.

$AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, $AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and U is 3×3 matrix whose columns are U_1, U_2, U_3 then answer the following questions

30. The value of |U| is

(A) 3
(B) –3
(C) 3/2
(D) 2

Sol. **(A)**

Let U_1 be $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ so that

$$\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

Similarly $U_2=\begin{bmatrix}2\\-1\\-4\end{bmatrix}$, $U_3=\begin{bmatrix}2\\-1\\-3\end{bmatrix}$.

Hence $U=\begin{bmatrix}1&2&2\\-2&-1&-1\\1&-4&-3\end{bmatrix}$ and $|U| = 3$.

31. The sum of the elements of U^{-1} is

(A) –1 (B) 0
(C) 1 (D) 3

Sol. **(B)**

Moreover adj $U=\begin{bmatrix}-1&-2&0\\-7&-5&-3\\9&6&3\end{bmatrix}$.

Hence $U^{-1}=\dfrac{\text{adj}U}{3}$ and sum of the elements of $U^{-1} = 0$.

32. The value of $\begin{bmatrix}3&2&0\end{bmatrix}U\begin{bmatrix}3\\2\\0\end{bmatrix}$ is

(A) 5 (B) 5/2
(C) 4 (D) 3/2

Sol. **(A)**

The value of $\begin{bmatrix}3&2&0\end{bmatrix}U\begin{bmatrix}3\\2\\0\end{bmatrix}$

$=\begin{bmatrix}3&2&0\end{bmatrix}\begin{bmatrix}1&2&2\\-2&-1&-1\\1&-4&-3\end{bmatrix}\begin{bmatrix}3\\2\\0\end{bmatrix}$

$=\begin{bmatrix}-1&4&4\end{bmatrix}\begin{bmatrix}3\\2\\0\end{bmatrix}=-3+8=5.$

Section – D

33. If roots of the equation $x^2 - 10cx - 11d = 0$ are a, b and those of $x^2 - 10ax - 11b = 0$ are c, d, then the value of $a + b + c + d$ is (a, b, c and d are distinct numbers)

Sol. As $a + b = 10c$ and $c + d = 10a$

$ab = -11d$, $cd = -11b$

$\Rightarrow$ $ac = 121$ and $(b + d) = 9(a + c)$

$a^2 - 10ac - 11d = 0$

$c^2 - 10ac - 11b = 0$

$\Rightarrow$ $a^2 + c^2 - 20ac - 11(b + d) = 0$

$\Rightarrow$ $(a + c)^2 - 22(121) - 11 \times 9(a + c) = 0$

$\Rightarrow$ $(a + c) = 121$ or -22 (rejected)

$\therefore$ $a + b + c + d = 1210.$

34. The value of $5050\dfrac{\int_0^1 (1-x^{50})^{100}dx}{\int_0^1 (1-x^{50})^{101}dx}$ is

Sol. $= \dfrac{5050\int_0^1 (1-x^{50})^{100}dx}{\int_0^1 (1-x^{50})^{101}dx} = 5050\dfrac{I_{100}}{I_{101}}$

$$I_{101} = \int_0^1 (1-x^{50})(1-x^{50})^{100}dx$$

$$= I_{100} - \int_0^1 x \cdot x^{49}(1-x^{50})^{100}dx$$

$$= I_{100} - \left[\frac{-x(1-x^{50})^{101}}{101}\right]_0^1 - \int_0^1 \frac{(1-x^{50})^{101}}{5050}$$

$$I_{101} = I_{100} - \frac{I_{101}}{5050}$$

$\Rightarrow$ $5050\dfrac{I_{100}}{I_{101}} = 5051.$

35. If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots (-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$, then find the minimum natural number n_0 such that $b_n > a_n \ \forall\ n > n_0$

Sol. $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots + (-1)^{x-1}\left(\frac{3}{4}\right)^n$

$$= \frac{\frac{3}{4}\left(1-\left(-\frac{3}{4}\right)^n\right)}{1+\frac{3}{4}} = \frac{3}{7}\left(1-\left(-\frac{3}{4}\right)^n\right)$$

$b_n > a_n \Rightarrow 2a_n < 1$

$\Rightarrow$ $\frac{6}{7}\left(1-\left(-\frac{3}{4}\right)^n\right) < 1$

$\Rightarrow$ $1-\left(-\frac{3}{4}\right)^n < \frac{7}{6}$

$\Rightarrow$ $-\frac{1}{6} < \left(-\frac{3}{4}\right)^n$ $\Rightarrow$ minimum natural number $n_0 = 6$.

36. If f(x) is a twice differentiable function such that f(a) = 0, f(b) = 2, f(c) = – 1, f(d) = 2, f(e) = 0, where a < b < c < d < e, then the minimum number of zeroes of $g(x) = (f'(x))^2 + f''(x)\,f(x)$ in the interval [a, e] is

Sol. $g(x) = \frac{d}{dx}\left(f(x) \cdot f'(x)\right)$

to get the zero of g(x) we take function

$h(x) = f(x) \,.\, f'(x)$

between any two roots of h(x) there lies at least one root of $h'(x) = 0$

$\Rightarrow$ g(x) = 0

$h(x) = 0$

$\Rightarrow$ $f(x) = 0$ or $f'(x) = 0$

$f(x) = 0$ has 4 minimum solutions

$f'(x) = 0$ minimum three solution

$h(x) = 0$ minimum 7 solution

$\Rightarrow$ $h'(x) = g(x) = 0$ has minimum 6 solutions.

Section – E

37. Match the following:

Normals are drawn at points P, Q and R lying on the parabola $y^2 = 4x$ which intersect at (3, 0). Then

(i)	Area of ΔPQR	(A) 2
(ii)	Radius of circumcircle of ΔPQR	(B) 5/2
(iii)	Centroid of ΔPQR	(C) (5/2, 0)
(iv)	Circumcentre of ΔPQR	(D) (2/3, 0)

Sol. As normal passes through (3, 0)

$\Rightarrow$ $0 = 3m - 2m - m^3$

$\Rightarrow$ $m^3 = m \Rightarrow m = 0, \pm 1$

$\therefore$ Centroid $\equiv \left(\frac{\left(m_1^2 + m_2^2 + m_3^2\right)}{3}, -\frac{2\left(m_1 + m_2 + m_3\right)}{3}\right) = \left(\frac{2}{3}, 0\right)$

Circum radius = $\left|\frac{-2m_1 + 2m_2}{2}\right| = 2$ units.

$Q \equiv \left(m_2^2, -2m_2\right) \equiv (1, -2)$

$R \equiv \left(m_3^2, -2m_3\right) \equiv (1, 2)$

Area of ΔPQR = $\frac{1}{2} \times 4 \times 1 = 2$ sq. units.

$R = \frac{QR}{2\sin\angle QPR} = \frac{4}{2\sin(2\tan^{-1}2)}$

$\Rightarrow$ $\frac{4}{2 \times \sin\left(\tan^{-1}\frac{4}{1-4}\right)} = \frac{4}{2 \times \frac{4}{5}} = \frac{5}{2}$

$\therefore$ circumcentre $\equiv \left(\frac{5}{2}, 0\right)$.

38. Match the following

(i) $\int_0^{\pi/2} (\sin x)^{\cos x}\left(\cos x \cot x - \log\left(\sin x\right)^{\sin x}\right)dx$ — (A) 1

(ii) Area bounded by $-4y^2 = x$ and $x - 1 = -5y^2$ — (B) 0

(iii) Cosine of the angle of intersection of curves $y = 3^{x-1}\log x$ and $y = x^x - 1$ is — (C) 6 ln 2

(iv) Data could not be retrieved. — (D) 4/3

Sol. (i) $I = \int_0^{\pi/2} (\sin x)^{\cos x}(\cos x \cdot \cot x - \log(\sin x)^{\sin x})dx$

$\Rightarrow$ $I = \int_0^{\pi/2} \frac{d}{dx}(\sin x)^{\cos x}\,dx = 1$.

(ii) The points of intersection of $-4y^2 = x$ and $x - 1 = -5y^2$ is (–4, –1) and (–4, 1)

Hence required area = $2\left[\left[\int_0^1 (1-5y^2)dy - \int_0^1 -4y^2 dy\right]\right] = \frac{4}{3}$.

(iii) The point of intersection of $y = 3^{x-1}\log x$ and $y = x^x - 1$ is (1, 0)

Hence $\frac{dy}{dx} = \frac{3^{x-1}}{x} + 3^{x-1}\log 3.\log x$. $\left.\frac{dy}{dx}\right|_{(1,\,0)} = 1$

for $y = x^x - 1$. $\left.\frac{dy}{dx}\right|_{(1,\,0)} = 1$

If θ is the angle between the curve then tanθ = 0 ⇒ cosθ = 1.

(iv) $\frac{dy}{dx} = \left(\frac{2}{x+y}\right)$

$\Rightarrow \quad \frac{dx}{dy} - \frac{x}{2} = \frac{y}{2}$

$\Rightarrow \quad xe^{-y/2} = \frac{1}{2}\int y \cdot e^{-y/2} dy$

$\Rightarrow \quad x + y + 2 = ke^{y/2} = 3e^{y/2}$.

39. Match the following

(i) Two rays in the first quadrant x + y = |a| and ax – y = 1 intersects each other in the interval $a \in (a_0, \infty)$, the value of a_0 is — (A) 2

(ii) Point (α, β, γ) lies on the plane x + y + z = 2. Let $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$, $\hat{k} \times (\hat{k} \times \vec{a}) = 0$, then γ = . — (B) 4/3

(iii) $\left|\int_0^1 (1-y^2)dy\right| + \left|\int_1^0 (y^2-1)dy\right|$ — (C) $\left|\int_0^1 \sqrt{1-x}\,dx\right| + \left|\int_{-1}^0 \sqrt{1+x}\,dx\right|$

(iv) If sinA sinB sinC + cosA cosB = 1, then the value of sinC = — (D) 1

Sol. (i) Solving the two equations of ray i.e. x + y = |a| and ax – y = 1

we get $x = \frac{|a|+1}{a+1} > 0$ and $y = \frac{|a|-1}{a+1} > 0$

when a + 1 > 0; we get a > 1 ∴ $a_0 = 1$.

(ii) We have $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k} \;\Rightarrow\; \vec{a}\cdot\hat{k} = \gamma$

Now; $\hat{k} \times (\hat{k} \times \hat{a}) = (\hat{k}\cdot\vec{a})\hat{k} - (\hat{k}\cdot\hat{k})\vec{a}$

$= \gamma\hat{k} - (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k})$

$= \alpha\hat{i} + \beta\hat{j} = \vec{0} \;\Rightarrow\; \alpha = \beta = 0$

As α + β + γ = 2 ⇒ γ = 2.

(iii) $\left|\int_0^1 (1-y^2)dy\right| + \left|\int_1^0 (y^2-1)dy\right|$

$= 2\int_0^1 (1-y^2)dy = \frac{4}{3}$

$\left|\int_0^1 \sqrt{1-x}\,dx\right| + \left|\int_{-1}^0 \sqrt{1+x}\,dx\right| = 2\int_0^1 \sqrt{1-x}\,dx$

$= 2\int_0^1 \sqrt{x}\,dx = 2\cdot\frac{2}{3}\cdot x^{3/2}\Big|_0^1 = \frac{4}{3}$.

(iv) sinA sinB sinC + cosA cosB ≤ sinA sinB + cosA cosB = cos(A – B)
$\Rightarrow$ cos(A – B) ≥ 1 $\Rightarrow$ cos(A – B) = 1 $\Rightarrow$ sinC = 1.

40. Match the following

(i) $\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{2i^2}\right) = t$, then tan t = (A) 0

(ii) Sides a, b, c of a triangle ABC are in AP and
$\cos\theta_1 = \frac{a}{b+c}$, $\cos\theta_2 = \frac{b}{a+c}$, $\cos\theta_3 = \frac{c}{a+b}$, then $\tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_3}{2}\right) =$ (B) 1

(iii) A line is perpendicular to x + 2y + 2z = 0 and passes through (0, 1, 0). (C) $\frac{\sqrt{5}}{3}$
The perpendicular distance of this line from the origin is
(D) 2/3

(iv) Data could not be retrieved.

Sol. (i) $\sum_{i=1}^{\infty} \tan^{-1}\left[\frac{1}{2i^2}\right] = t$

Now; $\sum_{i=1}^{\infty} \tan^{-1}\left[\frac{2}{4i^2 - 1 + 1}\right]$

$= \sum_{i=1}^{\infty}\left[\tan^{-1}(2i+1) - \tan^{-1}(2i-1)\right]$

$= \left[(\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}5 - \tan^{-1}3) + \cdots + \tan^{-1}(2n+1) - \tan^{-1}(2n-1)\ldots..\infty\right]$

$t = \tan^{-1}(2n+1) - \tan^{-1}1 = \lim_{n\to\infty} \tan^{-1}\frac{2n}{1+(2n+1)}$

$\Rightarrow \tan t = \lim_{n\to\infty}\frac{n}{n+1} \Rightarrow t = \frac{\pi}{4}$

(ii) We have $\cos\theta_1 = \frac{1-\tan^2\frac{\theta_1}{2}}{1+\tan^2\frac{\theta_1}{2}} = \frac{a}{b+c} \Rightarrow \tan^2\frac{\theta_1}{2} = \frac{b+c-a}{b+c+a}$

Also, $\cos\theta_3 = \frac{1-\tan^2\frac{\theta_3}{2}}{1+\tan^2\frac{\theta_3}{2}} = \frac{c}{a+b} \Rightarrow \tan^2\frac{\theta_3}{2} = \frac{a+b-c}{a+b+c}$

$\therefore \tan^2\frac{\theta_1}{2} + \tan^2\frac{\theta_3}{2} = \frac{2b}{3b} = \frac{2}{3}$

(iii) Line through (0, 1, 0) and perpendicular to plane x + 2y + 2z = 0 is given by $\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-1}{2} = r$.

Let P(r, 2r + 1, 2r) be the foot of perpendicular on the straight line then

r × 1 + (2r + 1) 2 + 2 × 2r = 0 $\Rightarrow r = -\frac{2}{9}$

$\therefore$ Point is given by $\left(-\frac{2}{9}, \frac{5}{9}, -\frac{4}{9}\right)$

$\therefore$ Required perpendicular distance = $\sqrt{\frac{4+25+16}{81}} = \frac{\sqrt{5}}{3}$ units.

(iv) Data could not be retrieved.