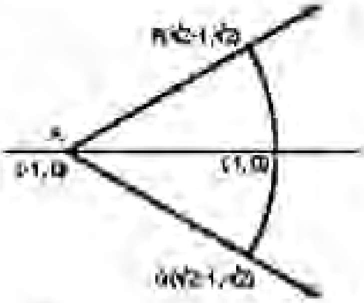
solutions to IIT–JEE, 2005 Screening

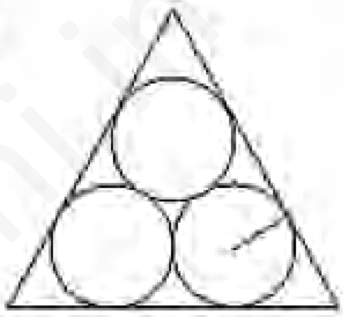
1. The locus of z which lies in shaded region is best represented by
(A) $z : |z + 1| > 2, |\arg(z + 1)| < \pi/4$
(B) $z : |z - 1| > 2, |\arg(z - 1)| < \pi/4$
(C) $z : |z + 1| < 2, |\arg(z + 1)| < \pi/2$
(D) $z : |z - 1| < 2, |\arg(z - 1)| < \pi/2$



Ans. A

Sol. The points $(1, 0)$, $\left(\sqrt{2}-1, -\sqrt{2}\right)$ and $\left(\sqrt{2}-1, \sqrt{2}\right)$ are equidistant from the point $(-1, 0)$.
The shaded area belongs to the region outside the sector of circle $|z + 1| = 2$, lying between the line rays $\arg(z + 1) = \frac{\pi}{4}$ and $\arg(z + 1) = \frac{-\pi}{4}$.

2. In an equilateral triangle, 3 coins of radii 1 unit each are kept so that they touch each other and also the sides of the triangle. Area of the triangle is
(A) $4 + 2\sqrt{3}$ (B) $6 + 4\sqrt{3}$
(C) $12 + \frac{7\sqrt{3}}{4}$ (D) $3 + \frac{7\sqrt{3}}{4}$



Ans. B

Sol. The line joining the vertex of the triangle and the centre of the coin makes angle $\frac{\pi}{6}$ with the sides of the triangle. The length of each of the sides of the equilateral triangle is $2 + 2\cot\frac{\pi}{6} = 2(1 + \sqrt{3})$.

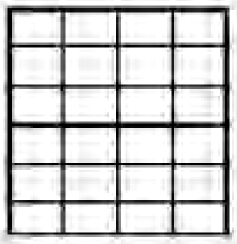
Hence its area is $\frac{\sqrt{3}}{4} 4(1 + \sqrt{3})^2 = 6 + 4\sqrt{3}$.

3. If a, b, c are integers not all equal and w is a cube root of unity ($w \neq 1$), then the minimum value of $|a + bw + cw^2|$ is
(A) 0 (B) 1
(C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{2}$

Ans. B

Sol. $|a + bw + cw^2| = \sqrt{\left(a - \frac{b}{2} - \frac{c}{2}\right)^2 + \frac{3}{4}(c - b)^2} = \sqrt{\frac{1}{2}\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)}$.
This is minimum when $a = b$ and $(b - c)^2 = (c - a)^2 = 1 \Rightarrow$ The minimum value is 1.

4. A rectangle with sides $2m - 1$ and $2n - 1$ is divided into squares of unit length by drawing parallel lines as shown in the diagram, then the number of rectangles possible with odd side lengths is
(A) $(m + n + 1)^2$ (B) 4^{m+n-1}
(C) m^2n^2 (D) $mn(m + 1)(n + 1)$



Ans. C

Sol There are 2m vertical (numbered 1, 2, ..., 2m) and 2n horizontal lines (numbered 1, 2, ..., 2n).
To form the required rectangle we must select two horizontal lines, one even numbered and one odd numbered and similarly two vertical lines. The number of rectangles is then ${}^mC_1 \cdot {}^mC_1 \cdot {}^nC_1 \cdot {}^nC_1 = m^2n^2$.

Alternate solution:
Number of rectangles possible is $(1 + 3 + 5 + \ldots + (2m - 1))(1 + 3 + 5 + \ldots + (2n - 1)) = m^2n^2$.

5. A circle is given by $x^2 + (y - 1)^2 = 1$, another circle C touches it externally and also the x-axis, then the locus of its centre is
(A) $\{(x, y) : x^2 = 4y\} \cup \{(x, y) : y \le 0\}$
(B) $\{(x, y) : x^2 + (y - 1)^2 = 4\} \cup \{(x, y) : y \le 0\}$
(C) $\{(x, y) : x^2 = y\} \cup \{(0, y) : y \le 0\}$
(D) $\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \le 0\}$

Ans. D

Sol Let the circle touching the x-axis be $x^2 + y^2 - 2ax - 2by + a^2 = 0$ with centre at (a, b) and radius b.
Since it touches the circle $x^2 + (y - 1)^2 = 1$, $|b + 1| = \sqrt{a^2 + (b - 1)^2}$
$\Rightarrow b^2 + 2b + 1 = a^2 + b^2 - 2b + 1$
$\Rightarrow 4b = a^2$ so that locus of (a, b) is $x^2 = 4y$. If the centre of the circle lies on the y-axis, then $y \le 0$

6. $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = 1/e$, where $\alpha, \beta \in [-\pi, \pi]$. Pairs of α, β which satisfy both the equations is/are
(A) 0
(B) 1
(C) 2
(D) 4

Ans. D

Sol For $\cos(\alpha - \beta) = 1$, $\alpha = \beta$ so that $\cos(\alpha + \beta) = 1/e \Rightarrow \alpha + \beta = \pm \cos^{-1} 1/e$
$\Rightarrow 2\alpha = \pm \cos^{-1}\left(\frac{1}{e}\right) \in [-2\pi, 2\pi]$. $\Rightarrow \alpha, \beta$ can be satisfied by 4 sets of values.

7. In ΔABC, a, b, c are the lengths of its sides and A, B, C are the angles of triangle ABC. The correct relation is given by
(A) $(b - c)\sin\left(\frac{B - C}{2}\right) = a\cos\frac{A}{2}$
(B) $(b - c)\cos\frac{A}{2} = a\sin\left(\frac{B - C}{2}\right)$
(C) $(b + c)\sin\left(\frac{B + C}{2}\right) = a\cos\frac{A}{2}$
(D) $(b - c)\cos\left(\frac{A}{2}\right) = 2a\sin\left(\frac{B + C}{2}\right)$

Ans B

Sol Here $\frac{b - c}{a} = \frac{\sin B - \sin C}{\sin A} = \frac{2\sin\frac{B - C}{2}\cos\frac{B + C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{\sin\left(\frac{B - C}{2}\right)}{\cos\frac{A}{2}}$

8. The value of $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} \ldots + \binom{30}{20}\binom{30}{30}$ is, where $\binom{n}{r} = {}^nC_r$
(A) $\binom{30}{10}$
(B) $\binom{30}{15}$
(C) $\binom{60}{30}$
(D) $\binom{31}{10}$

Ans. A

Sol The given expression is the coefficient of x^{20} in the product $(1 + x)^{30}(1 - x)^{30} = (1 - x^2)^{30}$
$\Rightarrow$ the given expression $= {}^{30}C_{10}$.

9. A variable plane at a distance of 1 unit from the origin cuts the co-ordinate axes at A, B and C. If the centroid D (x, y, z) of triangle ABC satisfies the relation $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k$, then the value of k is
(A) 3
(B) 1
(C) 1/3
(D) 9

Ans. D

Sol Let $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ be the variable plane so that $\left|\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=1$

The plane meets the coordinate axes at A (a, 0, 0), B (0, b, 0), C (0, 0, c). The centroid D of the triangle ABC is $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$

$\Rightarrow x=\frac{a}{3}, y=\frac{b}{3}, z=\frac{c}{3}$ and $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=1 \Rightarrow \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=9$.

10. If $\int_{\sin x}^{1} t^2(f(t))dt=(1-\sin x)$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is

(A) 1/3
(B) $1/\sqrt{3}$
(C) 3
(D) $\sqrt{3}$

Ans. C

Sol Differentiating both sides with respect to x, we get

$-\sin^2 x\, f(\sin x).\cos x=-\cos x \Rightarrow f(\sin x)=\frac{1}{\sin^2 x}$

$\Rightarrow f(x)=\frac{1}{x^2} \Rightarrow f\left(\frac{1}{\sqrt{3}}\right)=3$

11. In the quadratic equation $ax^2+bx+c=0$, if $\Delta=b^2-4ac$ and $\alpha+\beta, \alpha^2+\beta^2, \alpha^3+\beta^3$ are in G.P. where α, β are the roots of $ax^2+bx+c=0$, then

(A) $\Delta \neq 0$
(B) $b\Delta=0$
(C) $c\Delta=0$
(D) $\Delta=0$

Ans. C

Sol We have $(\alpha^2+\beta^2)^2=(\alpha+\beta)(\alpha^3+\beta^3) \Rightarrow \alpha\beta(\alpha-\beta)^2=0$

$\Rightarrow c\Delta=0$.

12. A six faced fair dice is thrown until 1 comes, then the probability that 1 comes in even no. of trials is

(A) 5/11
(B) 5/6
(C) 6/11
(D) 1/6

Ans. A

Sol The required probability $=\frac{1}{6}\cdot\frac{5}{6}+\frac{1}{6}\left(\frac{5}{6}\right)^3+\ldots$

$=\frac{1}{6}\cdot\frac{5}{6}\left[1+\left(\frac{5}{6}\right)^2+\ldots\right]=\frac{5}{11}$.

13. If f(x) is a twice differentiable function and given that f(1) = 1, f(2) = 4, f(3) = 9, then

(A) $f''(x)=2$, for $\forall\, x \in (1,3)$
(B) $f''(x)=f'(x)=5$ for some $x \in (2,3)$
(C) $f''(x)=3$, $\forall\, x \in (2,3)$
(D) $f''(x)=2$, for some $x \in (1,3)$

Ans. D

Sol Let $g(x)=f(x)-x^2$.

We have $g(1)=0, g(2)=0, g(3)=0$.

Hence by Rolle's theorem $g'(x)=0$ for some $c \in (1,2)$

and $g'(x)=0$ for some $d \in (2,3)$.

Again, by Rolle's theorem $g''(x)=0$ at some $x \in (c,d)$

$\Rightarrow f''(x)=2$ for some values $x \in (1,3)$

14. $\int_{-2}^{0}\left(x^3+3x^2+3x+3+(x+1)\cos(x+1)\right)dx$ is equal to

(A) −4
(B) 0
(C) 4
(D) 6

Ans. C

Sol. Here $I = \int_{-2}^{0} [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)]dx$

Put $x + 1 = t$

$= \int_{-1}^{1} \left[\left(t^3 + t\cos t\right) + 2\right]dt$

$= \int_{-1}^{1} 2dt = 4$.

15. If P (x) is a polynomial of degree less than or equal to 2 and S is the set of all such polynomials so that P (1) = 1, P (0) = 0 and P' (x) > 0 $\forall x \in [0, 1]$, then

(A) $S = \phi$

(B) $S = \{(1-a)x^2 + ax \quad 0 < a < 2\}$

(C) $(1-a)x^2 + ax \quad a \in (0, \infty)$

(D) $S = \{(1-a)x^2 + ax \quad 0 < a < 1\}$

Ans. B

Sol. Let the polynomial be $P(x) = ax^2 + bx + c$

$P(0) = 0 \Rightarrow c = 0$ and $P(1) = 1 \Rightarrow a + b = 1$ so that

$P'(x) = 2(1-b)x + b > 0 \; \forall x$

$\Rightarrow b \in (0, 2)$

$\Rightarrow S = \{(1-a)x^2 + ax, a \in (0, 2)\}$

16. The minimum area of triangle formed by the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and coordinate axes is

(A) ab sq. units

(B) $\frac{a^2 + b^2}{2}$ sq. units

(C) $\frac{(a+b)^2}{2}$ sq. units

(D) $\frac{a^2 + ab + b^2}{3}$ sq. units

Ans. A

Sol. A tangent of the given ellipse is $y = mx + \sqrt{a^2m^2 + b^2}$.

It meets the axes at $\left(\frac{-\sqrt{a^2m^2 + b^2}}{m}, 0\right)$ and $\left(0, \sqrt{a^2m^2 + b^2}\right)$.

Hence the area of the triangle is $\frac{1}{2}\left|\frac{a^2m^2 + b^2}{m}\right| = \frac{1}{2}\left|a^2m + \frac{b^2}{m}\right| \ge ab$

Alternate:

The equation of tangent at $(a\cos\theta, b\sin\theta)$ is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.

It meets the coordinate axes at $A \equiv (0, b\,\mathrm{cosec}\,\theta)$, $B \equiv (a\sec\theta, 0)$.

Area of triangle $= \frac{ab}{2\sin\theta\cos\theta} = \frac{ab}{\sin 2\theta} \ge ab$.

17. If the functions f(x) and g(x) are defined on $R \to R$ such that

$f(x) = \begin{cases} 0, & x \in \text{rational} \\ x, & x \in \text{irrational} \end{cases}$, $g(x) = \begin{cases} 0, & x \in \text{irrational} \\ x, & x \in \text{rational} \end{cases}$, then $(f - g)(x)$ is

(A) one-one and onto

(B) neither one-one nor onto

(C) one-one but not onto

(D) onto but not one-one

Ans. A

Sol. Let $h(x) = f(x) - g(x) = \begin{cases} x; & x \in \text{irrational} \\ -x; & x \in \text{rational} \end{cases}$

$\Rightarrow$ the function h(x) is one-one and onto.

18. The area bounded by the parabolas $y = (x+1)^2$ and $y = (x-1)^2$ and the line $y = 1/4$ is
(A) 4 sq. units
(B) 1/6 sq. units
(C) 4/3 sq. units
(D) 1/3 sq. units

Ans. D

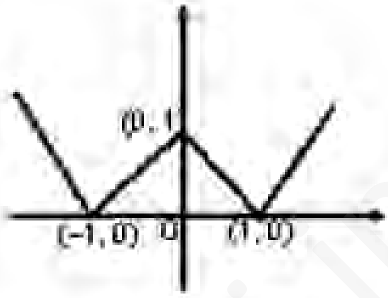
Sol. The parabolas meet at (0, 1) and intersect the line $y = 1/4$ at $x = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}$ and $\frac{3}{2}$.

Hence the required area $= 2\left[\int_0^{1/2}(x-1)^2\,dx\right] - \frac{1}{4} = \frac{2}{3}(x-1)^3\Big|_0^{1/2} - \frac{1}{4} = \frac{1}{3}$

19. The function given by $y = ||x| - 1|$ is differentiable for all real numbers except the points
(A) {0, 1, −1}
(B) ± 1
(C) 1
(D) −1

Ans. A

Sol. From the graph, the function is not differentiable at $x = -1, 0, 1$.



20. If $y = y(x)$ and it follows the relation $x\cos y + y\cos x = \pi$, then $y''(0)$
(A) 1
(B) − 1
(C) π
(D) - π

Ans. C

Sol. $x\cos y + y\cos x = \pi, y(0) = \pi$.

$\Rightarrow -x\sin y\frac{dy}{dx} + \cos y - y\sin x + \cos x\frac{dy}{dx} = 0 \Rightarrow y'(0) = 1$.

Again differentiating and using $y'(0) = 1$ and $y(0) = \pi$, we get $y''(0) = \pi$

21. The solution of primitive integral equation $(x^2 + y^2)\,dy = xy\,dx$, is $y = y(x)$. If $y(1) = 1$ and $y(x_0) = e$, then x_0 is
(A) $\sqrt{2(e^2 - 1)}$
(B) $\sqrt{2(e^2 + 1)}$
(C) $\sqrt{3}e$
(D) $\sqrt{\frac{e^2+1}{2}}$

Ans. C

Sol. We have $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$

Solving the homogenous differential equation by writing $y = vx$, we get

$-\frac{x^2}{2y^2} + \ln y = -\frac{1}{2}$.

For $y = e$, $\frac{-x_0^2}{2e^2} + \ln e = -\frac{1}{2} \Rightarrow x_0^2 = 3e^2 \Rightarrow x_0 = \sqrt{3}e$

22. $A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & -2 & 4\end{bmatrix}, I = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ and $A^{-1} = \left[\frac{1}{6}(A^2 + cA + dI)\right]$, then the value of c and d are
(A) - 6, - 11
(B) 6, 11
(C) - 6, 11
(D) 6, - 11

Ans. C

Sol We evaluate A^2 and A^3 and write the given equation as $AA^{-1} = I = \frac{1}{6}\left[A^2 + cA^{?} + dA\right]$

Comparing the corresponding elements on both the sides we get
$c = -6, d = 11$.
Alternatively, we may use Cayley Hamilton Theorem.

23. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$ and $x = P^T Q^{2005} P$, then x is equal to

(A) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{bmatrix}$

(C) $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$ (D) $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$

Ans A

Sol $P^T P = I$

$Q = PAP^T$ so that

$x = P^T Q^{2005} P = P^T (PAP^T)^{2005} P$

$= P^T PAP^T (PAP^T)^{2004} P$

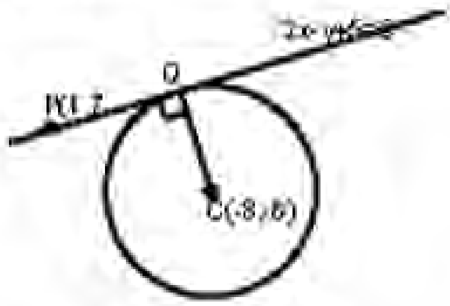
$= A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

24. Tangent to the curve $y = x^3 + 6$ at a point P (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the coordinates of Q are

(A) (−6, −11) (B) (−9, −13)
(C) (−10, −15) (D) (−6, −7)

Ans D

Sol Equation of tangent to the parabola at (1, 7) is



$x - \frac{(y+7)}{2} + 6 = 0 \Rightarrow 2x - y + 5 = 0$

$\Rightarrow$ Centre $\equiv (-8, -6)$

Equation of $CQ = x + 2y + k = 0$

$-8 - 12 + k = 0 \Rightarrow k = 20$

$PQ \equiv 4x - 2y + 10 = 0$

$CQ \equiv x + 2y + 20 = 0$

$= 5x + 30 = 0 \Rightarrow x = -6$

$\Rightarrow -6 + 2y + 20 = 0 \Rightarrow y = -7$

Hence the point of contact is (−6, −7).

25. If f(x) is a continuous and differentiable function and $f\left(\frac{1}{n}\right) = 0 \ \forall n \geq 1$ and $n \in I$, then

(A) $f(x) = 0, x \in (0, 1]$ (B) $f(0) = 0, f'(0) = 0$
(C) $f'(0) = 0 = f''(0), x \in (0, 1]$ (D) $f(0) = 0$ and $f'(0)$ need not to be zero

Ans B

Sol Given $f\left(\frac{1}{n}\right) = 0 \ \forall n \geq 1$ and $n \in I$.

This indicates that f(x) has a wavy behaviour.
Amplitude of the wave either (a) is constant (b) increases or (c) decreases.
In case of (a) and (b), function will not be differentiable at 0.
$\Rightarrow$ Amplitude has to decrease such that $f'(0) = 0$

26. If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero, non-coplanar vectors and $\vec{b}_1 = \vec{b} - \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$, $\vec{b}_2 = \vec{b} + \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}$,

$\vec{c}_1 = \vec{c} - \frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} + \frac{\vec{b}\cdot\vec{c}}{|\vec{c}|^2}\vec{b}_1$, $\vec{c}_2 = \vec{c} - \frac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2}\vec{a} - \frac{\vec{b}_1\cdot\vec{c}}{|\vec{b}_1|^2}\vec{b}_1$, $\vec{c}_3 = \vec{c} - \frac{\vec{c}\cdot\vec{a}}{|\vec{c}|^2}\vec{a} + \frac{\vec{b}\cdot\vec{c}}{|\vec{c}|^2}\vec{b}_1$, $\vec{c}_4 = \vec{c} - \frac{\vec{c}\cdot\vec{a}}{|\vec{c}|^2}\vec{a} - \frac{\vec{b}\cdot\vec{c}}{|\vec{b}|^2}\vec{b}_1$, then the set of orthogonal vectors is

(A) $(\vec{a}, \vec{b}_1, \vec{c}_3)$
(B) $(\vec{a}, \vec{b}_1, \vec{c}_2)$
(C) $(\vec{a}, \vec{b}_1, \vec{c}_1)$
(D) $(\vec{a}, \vec{b}_2, \vec{c}_2)$

Ans. B

Sol. Obviously $\vec{a}\cdot\vec{b}_1 = \left(\vec{b}\cdot\vec{a} - \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|^2}\vec{a}\cdot\vec{a}\right) = 0$

& $\vec{a}\cdot\vec{c}_2 = 0$ and $\vec{b}_1\cdot\vec{c}_2 = 0$

$\Rightarrow \left(\vec{a}, \vec{b}_1, \vec{c}_2\right)$ are orthogonal vectors.

27. For the primitive integral equation $y\,dx + y^2\,dy = x\,dy$; $x \in R$, $y > 0$, $y = y(x)$, $y(1) = 1$, then $y(-3)$ is
(A) 3
(B) 2
(C) 1
(D) 5

Ans. A

Sol. $y\frac{dx - x\,dy}{y^2} = -dy$

$\frac{x}{y} = -y + c$

$y(1) = 1 \Rightarrow c = 2$

$y^2 - 2y + x = 0$

$y(-3)$:

$y^2 - 2y - 3 = 0 \Rightarrow (y-3)(y+1) = 0$

$y = 3, -1$

28. X and Y are two sets and $f : X \to Y$. If $\{f(c) = y;\ c \subset X, y \subset Y\}$ and $\{f^{-1}(d) = x;\ d \subset Y, x \subset X\}$, then the true statement is
(A) $f(f^{-1}(b)) = b$
(B) $f^{-1}(f(a)) = a$
(C) $f(f^{-1}(b)) = b,\ b \subset y$
(D) $f^{-1}(f(a)) = a,\ a \subset x$

Ans. D

Sol. The given data is shown in the figure

Since $f^{-1}(d) = x$

$\Rightarrow f(x) = d$

Now, if $a \subset x$, $f(a) \subset d$

$\Rightarrow f^{-1}(f(a)) = a$