

1 RELATIONS AND FUNCTIONS

KEY CONCEPT INVOLVED

- 1. Relations** - Let A and B be two non-empty sets then every subset of $A \times B$ defines a relation from A to B and every relation from A to B is a subset of $A \times B$.
Let $R \subseteq A \times B$ and $(a, b) \in R$. then we say that a is related to b by the relation R as aRb . If $(a, b) \notin R$ as $a \not R b$.
- 2. Domain and Range of a Relation** - Let R be a relation from A to B, that is, let $R \subseteq A \times B$. then *Domain* $R = \{a : a \in A, (a, b) \in R \text{ for some } b \in B\}$ i.e. dom. R is the set of all the first elements of the ordered pairs which belong to R. *Range* $R = \{b : b \in B, (a, b) \in R \text{ for some } a \in A\}$ i.e. range R is the set of all the second elements of the ordered pairs which belong to R. Thus Dom. $R \subseteq A$, Range $R \subseteq B$.
- 3. Inverse Relation** - Let $R \subseteq A \times B$ be a relation from A to B. Then inverse relation $R^{-1} \subseteq B \times A$ is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$
It is clear that
 - $aRb = bR^{-1}a$
 - dom. $R^{-1} = \text{range } R$ and range $R^{-1} = \text{dom } R$.
 - $(R^{-1})^{-1} = R$.
- 4. Composition of Relation** - Let $R \subseteq A \times B$, $S \subseteq B \times C$ be two relations. Then composition of the relations R and S is denoted by $\text{SoR} \subseteq A \times C$ and is defined by $(a, c) \in (\text{SoR})$ iff $b \in B$ such that $(a, b) \in R$, $(b, c) \in S$.
- 5. Relations in a set** - let $A (\neq \phi)$ be a set and $R \subseteq A \times A$ i.e. R is a relation in the set A.
- 6. Reflexive Relations** - R is a reflexive relation if $(a, a) \in R, \forall a \in A$ it should be noted that if for any $a \in A$ such that $a \not R a$. then R is not reflexive.
- 7. Symmetric Relation** - R is called symmetric relation on A if $(x, y) \in R \Rightarrow (y, x) \in R$.
i.e. if x is related to y, then y is also related to x.
It should be noted that R is symmetric iff $R^{-1} = R$.
- 8. Anti Symmetric Relations** - R is called an anti symmetric relation if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$.
Thus if $a \neq b$ then a may be related to b or b may be related to a but never both.
- 9. Transitive Relations** - R is called a transitive relation if $(a, b) \in R$, $(b, c) \in R \Rightarrow (a, c) \in R$
- 10. Identity Relations** - R is an identity relation if $(a, b) \in R$ iff $a = b$. i.e. every element of A is related to only itself and always identity relation is reflexive symmetric and transitive.
- 11. Equivalence Relations** - a relation R in a set A is called an equivalence relation if
 - R is reflexive i.e. $(a, a) \in R \forall a \in A$
 - R is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$
 - R is transitive i.e. $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$.
- 12. Functions** - Suppose that to each element in a set A there is assigned, by some rule, an unique element of a set B. Such rules are called functions. If we let f denote these rules, then we write $f : A \rightarrow B$ as f is a function of A into B.
- 13. Equal Functions** - If f and g are functions defined on the same domain A and if $f(a) = g(a)$ for every $a \in A$, then $f = g$.

14. **Constant Functions** - Let $f: A \rightarrow B$. If $f(a) = b$, a constant, for all $a \in A$, then f is called a constant function. Thus f is called a constant function if range f consists of only one element.
15. **Identity Functions** - A function f is such that $A \rightarrow A$ is called an identity function iff $f(x) = x, \forall x \in A$ it is denoted by I_A .
16. **One-One Functions (Injective)** - Let $f: A \rightarrow B$ then f is called a one-one function. If no two different elements in A have the same image i.e. different elements in A have different elements in B .
Denoted by symbol f is one-one if

$$f(a) = f(a') \Rightarrow a = a'$$
i.e. $a \neq a' \Rightarrow f(a) \neq f(a')$
A mapping which is not one-one is called many one function.
17. **Onto functions (Surjective)** - In the mapping $f: A \rightarrow B$, if every member of B appears as the image of atleast one element of A , then we say “ f is a function of A onto B or simply f is an onto functions” Thus f is onto iff $f(A) = B$
i.e. range = codomain
A function which is not onto is called into function.
18. **Inverse of a function** - Let $f: A \rightarrow B$ and $b \in B$ then the inverse of b i.e. $f^{-1}(b)$ consists of those elements in A which are mapped onto b i.e. $f^{-1}(b) = \{x; x \in A, f(x) \in b\}$
 $\therefore f^{-1}(b) \subset A, f^{-1}(b)$ may be a null set or a singleton.
19. **Inverse Functions** - Let $f: A \rightarrow B$ be a one-one onto-function from A onto B . Then for each $b \in B, f^{-1}(b) \in A$ and is unique. So, $f^{-1}: B \rightarrow A$ is a function defined by $f^{-1}(b) = a$, iff $f(a) = b$.
Then f^{-1} is called the inverse function of f . If f has inverse function, f is also called invertible or non-singular.
Thus f is invertible (non-singular) iff it is one-one onto (bijective) function.
20. **Composition Functions** - Let $f: A \rightarrow B$ and $g: B \rightarrow C$, be two functions,
Then composition of f and g denoted by $g \circ f: A \rightarrow C$ is defined by $(g \circ f)(a) = g\{f(a)\}$.
21. **Binary Operation** - A binary operation $*$ on a set A is a function $*$: $A \times A \rightarrow A$. We denote $*(a, b)$ by $a * b$
22. **Commutative Binary Operation** - A binary operation $*$ on the set A is commutative if for every $a, b \in A, a * b = b * a$.
23. **Associative Binary Operation** - A binary operation $*$ on the set A is associative if $(a * b) * c = a * (b * c)$.
24. **An Identity Element e for Binary Operation** - Let $*$: $A \times A \rightarrow A$ be a binary operation. There exists an element $e \in A$ such that $a * e = a = e * a \forall a \in A$, then e is called an identity element for Binary Operation $*$.
25. **Inverse of an Element a** - Let $*$: $A \times A \rightarrow A$ be a binary operation with identity element e in A . an element $a \in A$ is invertible w.r.t. binary operation $*$, if there exists an element b in A such that $a * b = e = b * a$. and b is called the inverse of a and is denoted by a^{-1} .

CONNECTING CONCEPTS

1. In general $g \circ f \neq f \circ g$.
2. $f: A \rightarrow B$, be one-one, onto then
 $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$
3. $f: A \rightarrow B, g: B \rightarrow C, h: C \rightarrow D$
then $(h \circ g) \circ f = h \circ (g \circ f)$.
4. $f: A \rightarrow B, g: B \rightarrow C$ be one-one and onto then $g \circ f: A \rightarrow C$ is also one-one onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
5. Let $f: A \rightarrow B$, then $f \circ I_B = f$ and $f \circ I_A = f$. It should be noted that $f \circ I_B$ is not defined since for $(f \circ I_B)(x) = f \circ \{I_B(x)\} = f(x)$
 $I_B(x)$ exist when $x \in B$ and $f(x)$ exist when $x \in A$
6. $f: A \rightarrow B, g: B \rightarrow C$ are both one-one, then $g \circ f: A \rightarrow C$ is also one-one it should be noted that for $g \circ f$ to be one-one f must be one-one.
7. If $f: A \rightarrow B, g: B \rightarrow C$ are both onto then $g \circ f$ must be onto. However, the converse is not true. But for $g \circ f$ to be onto g must be onto.

8. The domain of the functions

$$(f+g)(x) = f(x) + g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x)$$

is given by $(\text{dom. } f) \cap (\text{dom. } g)$ while domain of the function $(f/g)(x) = \frac{f(x)}{g(x)}$ is given by.

$$(\text{dom. } f) \cap (\text{dom. } g) - \{x : g(x) = 0\}$$

9. If $O(A) = m$, $O(B) = n$, then total number of mappings from A to B is n^m .
10. If A and B are finite sets and $O(A) = m$, $O(B) = n$, $m \leq n$.

Then number of injection (one-one) from A to B is ${}^n P_m = \frac{n!}{(n-m)!}$

11. If $f: A \rightarrow B$ is injective (one-one), then $O(A) \leq O(B)$.
12. If $f: A \rightarrow B$ is surjective (onto), then $O(A) \geq O(B)$.
13. If $f: A \rightarrow B$ is bijective (one-one onto), then $O(A) = O(B)$.
14. Let $f: A \rightarrow B$ and $O(A) = O(B)$, then f is one-one \Leftrightarrow it is onto.
15. Let $f: A \rightarrow B$ and $X_1, X_2 \subseteq A$, then f is one-one iff $f(X_1 \cap X_2) = f(X_1) \cap f(X_2)$
16. Let $f: A \rightarrow B$ and $X \subseteq A, Y \subseteq B$, then in general $f^{-1}(f(X)) \subseteq X$, $f(f^{-1}(Y)) \subseteq Y$
If f is one-one onto $f^{-1}(f(X)) = X$, $f(f^{-1}(Y)) = Y$.