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## VECTOR ALGEBRA

### KEY CONCEPT INVOLVED

- Vector** – A vector is a quantity having both magnitude and direction, such as displacement, velocity, force and acceleration.

$\overline{AB}$  is a directed line segment. It is a vector  $\overline{AB}$  and its direction is from A to B.

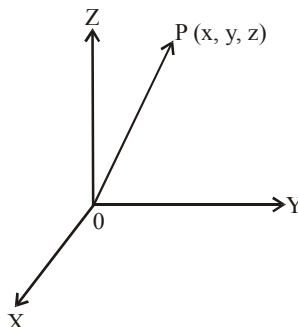


**Initial Points** – The point A where from the vector  $\overline{AB}$  starts is known as initial point.

**Terminal Point** – The point B, where it ends is said to be the terminal point.

**Magnitude** – The distance between initial point and terminal point of a vector is the magnitude or length of the vector  $\overline{AB}$ . It is denoted by  $|\overline{AB}|$  or AB.

- Position Vector** – Consider a point p (x, y, z) in space. The vector  $\overline{OP}$  with initial point, origin O and terminal point P, is called the position vector of P.



### 3. Types of Vectors

(i) **Zero Vector Or Null Vector** – A vector whose initial and terminal points coincide is known as zero vector ( $\vec{0}$ ).

(ii) **Unit Vector** – A vector whose magnitude is unity is said to be unit vector. It is denoted as  $\hat{a}$  so that  $|\hat{a}| = 1$ .

(iii) **Co-initial Vectors** – Two or more vectors having the same initial point are called co-initial vectors.

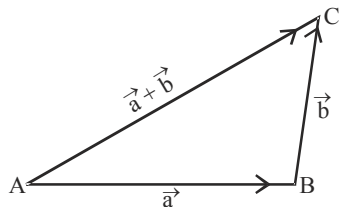
(iv) **Collinear Vectors** – If two or more vectors are parallel to the same line, such vectors are known as collinear vectors.

(v) **Equal Vectors** – If two vectors  $\vec{a}$  and  $\vec{b}$  have the same magnitude and direction regardless of the positions of their initial points, such vectors are said to be equal *i.e.*,  $\vec{a} = \vec{b}$ .

(vi) **Negative of a vector** – A vector whose magnitude is same as that of a given vector  $\overline{AB}$ , but the direction is opposite to that of it, is known as negative of vector  $\overline{AB}$  *i.e.*,  $\overline{BA} = -\overline{AB}$

### 4. Sum of Vectors

(i) **Sum of vectors  $\vec{a}$  and  $\vec{b}$**  let the vectors  $\vec{a}$  and  $\vec{b}$  be so positioned that initial point of one coincides with terminal point of the other. If  $\vec{a} = \overline{AB}$ ,  $\vec{b} = \overline{BC}$ . Then the vector  $\vec{a} + \vec{b}$  is represented by the third side of  $\Delta ABC$ . *i.e.*,  $\overline{AB} + \overline{BC} = \overline{AC}$  ... (i)



This is known as the triangle law of vector addition.

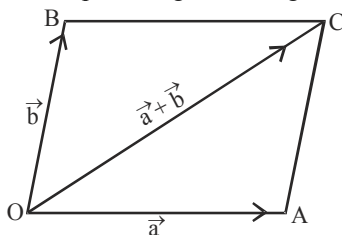
Further  $\overline{AC} = -\overline{CA}$

$$\overline{AB} + \overline{BC} = -\overline{CA} \quad \therefore \overline{AB} + \overline{BC} + \overline{CA} = 0$$

when sides of a triangle ABC are taken in order i.e. initial and terminal points coincides. Then

$$\overline{AB} + \overline{BC} + \overline{CA} = 0$$

- (ii) **Parallelogram law of vector addition** – If the two vectors  $\vec{a}$  and  $\vec{b}$  are represented by the two adjacent sides OA and OB of a parallelogram OACB, then their sum  $\vec{a} + \vec{b}$  is represented in magnitude and direction by the diagonal OC of parallelogram through their common point O i.e.,  $\overline{OA} + \overline{OB} = \overline{OC}$

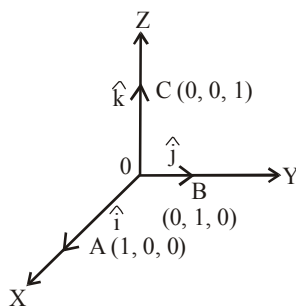


5. **Multiplication of Vector by a Scalar** – Let  $\vec{a}$  be the given vector and  $\lambda$  be a scalar, then product of  $\lambda$  and  $\vec{a} = \lambda\vec{a}$

(i) when  $\lambda$  is +ve, then  $\vec{a}$  and  $\lambda\vec{a}$  are in the same direction.

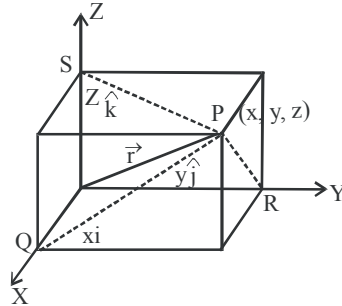
(ii) when  $\lambda$  is -ve. then  $\vec{a}$  and  $\lambda\vec{a}$  are in the opposite direction. Also  $|\lambda\vec{a}| = |\lambda| |\vec{a}|$ .

6. **Components of Vector** – Let us take the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) on the coordinate axes OX, OY and OZ respectively. Now,  $|\overline{OA}| = 1$ ,  $|\overline{OB}| = 1$  and  $|\overline{OC}| = 1$ , Vectors  $\overline{OA}$ ,  $\overline{OB}$  and  $\overline{OC}$  each having magnitude 1 is known as unit vector. These are denoted by  $\hat{i}$ ,  $\hat{j}$  and  $\hat{k}$ .



Consider the vector  $\overline{OP}$ , where P is the point (x, y, z). Now OQ, OR, OS are the projections of OP on coordinates axes.

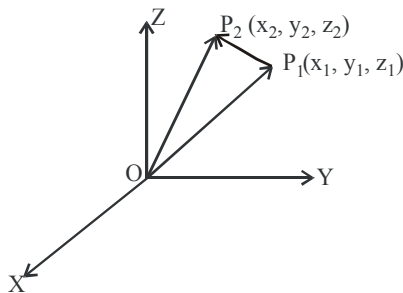
$$\therefore OQ = x, OR = y, OS = z \quad \therefore \overline{OQ} = x\hat{i}, \overline{OR} = y\hat{j}, \overline{OS} = z\hat{k}$$



$$\Rightarrow \quad \overline{OP} = x\hat{i} + y\hat{j} + z\hat{k} \quad , \quad |\overline{OP}| = \sqrt{x^2 + y^2 + z^2} = |\vec{r}|$$

$x, y, z$  are called the scalar components and  $x\hat{i}, y\hat{j}, z\hat{k}$  are called the vector components of vector  $\overline{OP}$ .

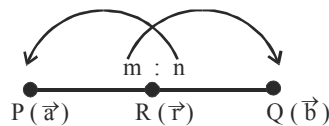
7. **Vector joining two points** – Let  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$  be the two points. Then vector joining the points  $P_1$  and  $P_2$  is  $\overline{P_1P_2}$ . Join  $P_1, P_2$  with  $O$ . Now  $\overline{OP_2} = \overline{OP_1} + \overline{P_1P_2}$  (by triangle law)



$$\begin{aligned} \therefore \quad \overline{P_1P_2} &= \overline{OP_2} - \overline{OP_1} \\ &= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \\ |\overline{P_1P_2}| &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \end{aligned}$$

### 8. Section Formula

- (i) A line segment  $PQ$  is divided by a point  $R$  in the ratio  $m : n$  internally *i.e.*,  $\frac{PR}{RQ} = \frac{m}{n}$

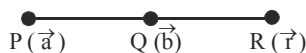


If  $\vec{a}$  and  $\vec{b}$  are the position vectors of  $P$  and  $Q$  then the position vector  $\vec{r}$  of  $R$  is given by

$$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$$

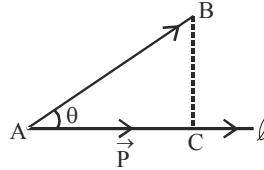
If  $R$  be the mid-point of  $PQ$ , then  $\vec{r} = \frac{\vec{a} + \vec{b}}{2}$

- (ii) when  $R$  divides  $PQ$  externally, *i.e.*,  $|\vec{a}| \cdot |\vec{b}| \hat{n}$



$$\text{Then } \vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

- 9. Projection of vector along a directed line** – Let the vector  $\overline{AB}$  makes an angle  $\theta$  with directed line  $\ell$ .  
Projection of AB on  $\ell = |\overline{AB}| \cos \theta = \overline{AC} = \vec{p}$ .



The vector  $\vec{p}$  is called the projection vector. Its magnitude is  $|\vec{p}|$ , which is known as projection of vector  $\overline{AB}$ . The angle  $\theta$  between  $\overline{AB}$  and  $\overline{AC}$  is given by

$$\begin{aligned} \cos \theta &= \frac{\overline{AB} \cdot \overline{AC}}{|\overline{AB}| |\overline{AC}|}, \quad \text{Now projection } AC = |\overline{AB}| \cos \theta = \frac{\overline{AB} \cdot \overline{AC}}{|\overline{AC}|} \\ &= \overline{AB} \cdot \left( \frac{\overline{AC}}{|\overline{AC}|} \right), \quad \text{If } \overline{AB} = \vec{a}, \text{ then } \overline{AC} = \vec{a} \cdot \left( \frac{\vec{p}}{|\vec{p}|} \right) = \vec{a} \cdot \hat{p} \end{aligned}$$

Thus, the projection of  $\vec{a}$  on  $\vec{b} = \vec{a} \cdot \left( \frac{\vec{b}}{|\vec{b}|} \right) = \vec{a} \cdot \hat{b}$

- 10. Scalar Product of Two Vectors (Dot Product)** – Scalar Product of two vectors  $\vec{a}$  and  $\vec{b}$  is defined as  
 $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$

Where  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$  ( $0 \leq \theta \leq \pi$ )

(i) when  $\theta = 0$ , then  $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| = ab$  Also  $\vec{a} \cdot \vec{a} = |\vec{a}| |\vec{a}| = a \cdot a = a^2$

$$\therefore \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

(ii) when  $\theta = \frac{\pi}{2}$ , then  $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \frac{\pi}{2} = 0$

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

- 11. Vector Product of two Vectors (Cross Product)** – The vector product of two non-zero vectors  $\vec{a}$  and  $\vec{b}$ , denoted by  $\vec{a} \times \vec{b}$  is defined as

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \cdot \hat{n}, \quad \text{where } \theta \text{ is the angle between } \vec{a} \text{ and } \vec{b}, 0 \leq \theta \leq \pi.$$

Unit vector  $\hat{n}$  is perpendicular to both vectors  $\vec{a}$  and  $\vec{b}$  such that  $\vec{a} \cdot \vec{b}$  and  $\hat{n}$  form a right handed orthogonal system.

(i) If  $\theta = 0$ , then  $\vec{a} \times \vec{b} = 0$ ,  $\therefore \vec{a} \times \vec{a} = 0$

$$\text{and } \therefore \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

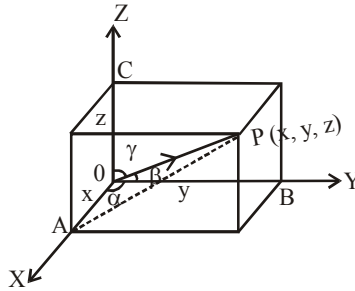
(ii) If  $\theta = \frac{\pi}{2}$ , then  $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \hat{n}$

$$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$$

$$\text{Also, } \hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i} \text{ and } \hat{i} \times \hat{k} = -\hat{j}$$

## CONNECTING CONCEPTS

- 1. Direction Cosines** – Let OX, OY, OZ be the positive coordinate axes, P(x, y, z) by any point in the space. Let  $\overline{OP}$  makes angles  $\alpha, \beta, \gamma$  with coordinate, axes OX, OY, OZ. The angle  $\alpha, \beta, \gamma$  are known as direction angles, cosine of these angles *i.e.*,



$\cos \alpha, \cos \beta, \cos \gamma$  are called direction cosines of line OP. these direction cosines are denoted by  $\ell, m, n$  *i.e.*,  $\ell = \cos \alpha, m = \cos \beta, n = \cos \gamma$

- 2. Relation Between,  $\ell, m, n$  and Direction Ratios** –

The perpendiculars PA, PB, PC are drawn on coordinate axes OX, OY, OZ respectively. Let  $|\overline{OP}| = r$

In  $\triangle OAP$ ,  $\angle A = 90^\circ$ ,  $\cos \alpha = \frac{x}{r} = \ell$ ,  $\therefore x = \ell r$ , In  $\triangle OBP$ ,  $\angle B = 90^\circ$ ,  $\cos \beta = \frac{y}{r} = m$   $\therefore y = mr$

In  $\triangle OCP$ ,  $\angle C = 90^\circ$ ,  $\cos \gamma = \frac{z}{r} = n$ ,  $\therefore z = nr$

Thus the coordinates of P may be expressed as  $(\ell r, mr, nr)$

Also,  $OP^2 = x^2 + y^2 + z^2, r^2 = (\ell r)^2 + (mr)^2 + (nr)^2 \Rightarrow \ell^2 + m^2 + n^2 = 1$

Set of any three numbers, which are proportional to direction cosines are called direction ratio of the vector. Direction ratio are denoted by a, b and c.

The numbers  $\ell r, mr$  and  $nr$ , proportional to the direction cosines, hence, they are also direction ratios of vector  $\overline{OP}$ .

- 3. Properties of Vector Addition** –

1. For two vectors  $\vec{a}, \vec{b}$  the sum is commutative *i.e.*,  $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

2. For three vectors  $\vec{a}, \vec{b}$  and  $\vec{c}$ , the sum of vectors is associative *i.e.*,

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

4. **Additive Inverse of Vector  $\vec{a}$**  – If there exists vector  $-\vec{a}$  such that  $\vec{a} + (-\vec{a}) = \vec{a} - \vec{a} = \vec{0}$  then  $-\vec{a}$  is called the additive inverse of  $\vec{a}$

5. **Some Properties** – Let  $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$  and  $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$

(i)  $\vec{a} + \vec{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) + (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) = (a_1 + b_1) \hat{i} + (a_2 + b_2) \hat{j} + (a_3 + b_3) \hat{k}$

(ii)  $\vec{a} = \vec{b}$  or  $(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) = (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) \Rightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3$

(iii)  $\lambda \vec{a} = \lambda (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) = (\lambda a_1) \hat{i} + (\lambda a_2) \hat{j} + (\lambda a_3) \hat{k}$

(iv)  $\vec{a}$  and  $\vec{b}$  are parallel, if and only if there exists a non zero scalar  $\lambda$  such that  $\vec{b} = \lambda \vec{a}$

$$\text{i.e., } b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k} = \lambda (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) = (\lambda a_1) \hat{i} + (\lambda a_2) \hat{j} + (\lambda a_3) \hat{k}$$

$$\therefore b_1 = \lambda a_1, \quad b_2 = \lambda a_2, \quad b_3 = \lambda a_3 \quad \therefore \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

### 6. Properties of scalar product of two vectors (Dot Product)

$$(i) \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

$$\text{If } \vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \text{ and } \vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$\text{Then, } \vec{a} \cdot \vec{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}), \quad \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}, \quad |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2} \quad \therefore \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}}$$

$$(ii) \vec{a} \cdot \vec{b} \text{ is commutative } \text{i.e., } \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$(iii) \text{ If } \alpha \text{ is a scalar, then } (\alpha \vec{a}) \cdot \vec{b} = \alpha (\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\alpha \vec{b})$$

### 7. Properties of Vector Product of two Vectors (Cross Product)–

$$(i) (a) \text{ If } \vec{a} = 0 \text{ or } \vec{b} = 0, \text{ then } \vec{a} \times \vec{b} = 0$$

$$(b) \text{ If } \vec{a} \parallel \vec{b}, \text{ then } \vec{a} \times \vec{b} = 0$$

$$(ii) \vec{a} \times \vec{b} \text{ is not commutative}$$

$$\text{i.e. } \vec{a} \times \vec{b} = \vec{b} \times \vec{a}, \text{ but } \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$(iii) \text{ If } \vec{a} \text{ and } \vec{b} \text{ represent adjacent sides of a parallelogram, then its area } |\vec{a} \times \vec{b}|$$

$$(iv) \text{ If } \vec{a}, \vec{b} \text{ represent the adjacent sides of a triangle, then its area} = \frac{1}{2} |\vec{a} \times \vec{b}|$$

$$(v) \text{ Distributive property } \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$(a) \text{ If } \alpha \text{ be a scalar, then } \alpha (\vec{a} \times \vec{b}) = (\alpha \vec{a}) \times \vec{b} = \vec{a} \times (\alpha \vec{b})$$

$$(b) \text{ If } \vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}, \text{ and } \vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$$

$$\text{Then, } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

8. If  $\alpha_1, \beta_1, \gamma$  are the direction angles of the vector  $\vec{a} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k})$ . Then direction cosines of  $\vec{a}$  are given as

$$\cos \alpha = \frac{a_1}{|\vec{a}|}, \quad \cos \beta = \frac{a_2}{|\vec{a}|}, \quad \cos \gamma = \frac{a_3}{|\vec{a}|}$$

9. **Scalar Product of Two Vectors (Dot Product)** – Scalar Product of two vectors  $\vec{a}$  and  $\vec{b}$  is defined as

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

where  $\theta$  is the angle between  $\vec{a}$  and  $\vec{b}$   $\left(0 \leq \theta < \frac{\pi}{2}\right)$

(i) When  $\theta = 0$ , then  $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}|$ . Also  $\vec{a} \cdot \vec{a} = a \cdot a = a^2$

$$\therefore \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$

(ii) When  $\theta = \frac{\pi}{2}$ ,  $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \frac{\pi}{2} = 0$