

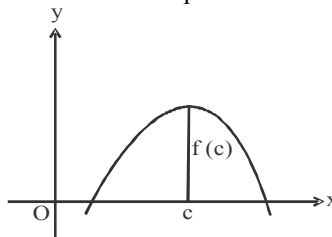


# 6

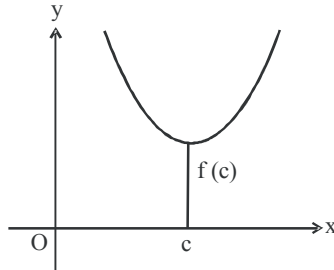
## APPLICATION OF DERIVATIVES

### KEY CONCEPTS INVOLVED

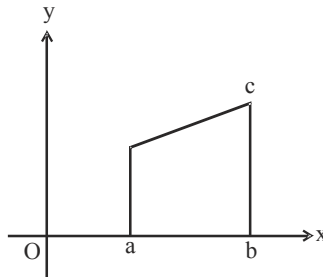
- 1. Rate of change of Quantities** – Let  $y = f(x)$  be a function. If the change in one quantity  $y$  varies with another quantity  $x$ , then  $\frac{dy}{dx}$  or  $f'(x)$  denotes the rate of change of  $y$  with respect to  $x$ .  $\left. \frac{dy}{dx} \right|_{x=x_0}$  or  $f'(x_0)$  represents the rate of change of  $y$  w.r.t.  $x$  at  $x = x_0$ .
- 2. Increasing and Decreasing function at  $x_0$**  A function  $f$  is said to be
  - (a) Increasing on an interval  $(a, b)$  if  $x_1 < x_2$  in  $(a, b) \Rightarrow f(x_1) < f(x_2)$  for all  $x_1, x_2 \in (a, b)$   
Alternatively, if  $f'(x) \geq 0$  for each  $x$  in  $(a, b)$
  - (b) Decreasing on  $(a, b)$  if  $x_1 < x_2$  in  $(a, b) \Rightarrow f(x_1) \geq f(x_2)$  for all  $x_1, x_2 \in (a, b)$  Alternatively, if  $f'(x) \leq 0$  for each  $x$  in  $(a, b)$
- 3. Test : Increasing/decreasing/constant function** – Let  $f$  be a continuous on  $[a, b]$  and differentiable in an open interval  $(a, b)$ , then.
  - (i)  $f$  is increasing on  $[a, b]$ , if  $f'(x) > 0$  for each  $x \in (a, b)$
  - (ii)  $f$  is decreasing on  $[a, b]$ , if  $f'(x) < 0$  for each  $x \in (a, b)$
  - (iii)  $f$  is constant on  $[a, b]$ , if  $f'(x) = 0$  for each  $x \in (a, b)$
- 4. Tangent to a Curve** – Let  $y = f(x)$  be the equation of a curve. The equation of the tangent at  $(x_0, y_0)$  is  $y - y_0 = m(x - x_0)$ , where  $m = \text{slope of the tangent} = \left. \frac{dy}{dx} \right|_{(x_0, y_0)}$  or  $f'(x_0)$
- 5. Normal to the Curve** – Let  $y = f(x)$  be the equation of the curve Equation of the normal at  $(x_0, y_0)$  is
$$y - y_0 = -\frac{1}{m}(x - x_0)$$
where
$$m = \text{Slope of the tangent at } (x_0, y_0) = \left. \frac{dy}{dx} \right|_{(x_0, y_0)} \text{ or } f'(x_0)$$
- 6. Approximation** – Let  $y = f(x)$ ,  $\Delta x$  be a small increment in  $x$  and  $\Delta y$  be the increment in  $y$  corresponding to the increment in  $x$ , i.e.,  $\Delta y = f(x + \Delta x) - f(x)$ . Then approximate value of  $\Delta y = \left( \frac{dy}{dx} \right) \Delta x$
- 7. Maximum Value, Minimum value, Extreme Value** – Let  $f$  be a function defined in the interval  $I$ , then
  - (i) **Maximum Value** – If there exists a point  $c$  in  $I$  such that  $f(c) \geq f(x)$ , for all  $x \in I$  then  $f(c)$  is called maximum value of  $f$  in  $I$ . The point  $c$  is known as a point of maximum value of  $f$  in  $I$ .



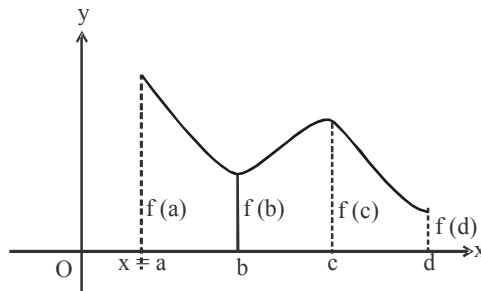
- (ii) **Minimum Value** – If there exists a point  $c$  in  $I$  such that  $f(c) \leq f(x)$ ,  $\forall x \in I$ , then  $f(c)$  is called the minimum value of  $f$  in  $I$ . The point  $c$  is called a point of minimum value of  $f$  in  $I$ .



- (iii) **Extreme Value** – If there exists a point  $c$  in  $I$  such that  $f(c)$  is either a maximum value or a minimum value of  $f$  in  $I$ , then  $f(c)$  is the extreme value of  $f(x)$  in  $I$ . The point  $c$  is said to be an extreme point.



8. **Absolute Maxima and Minima** – let  $f$  be a continuous function on an interval  $I = [a, b]$ . Then  $f$  has the absolute maximum value and  $f$  attains it at least once in  $I$ . Similarly,  $f$  has the absolute minimum value and attains it at least once in  $I$ .



At  $x = b$ , there is a local minima

At  $x = c$ , there is a local maxima

At  $x = a$ ,  $f(a)$  is the greatest value or absolute max. value.

At  $x = d$ ,  $f(d)$  is the least value or absolute min. value.

9. **Local Maxima and Minima** – let  $f$  be a real valued function and  $c$  be an interior point in the domain of  $f$ , then

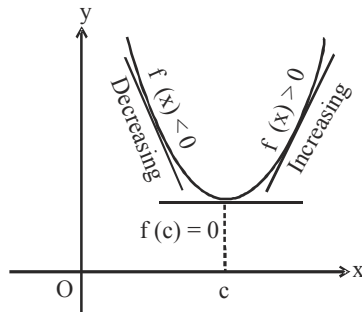
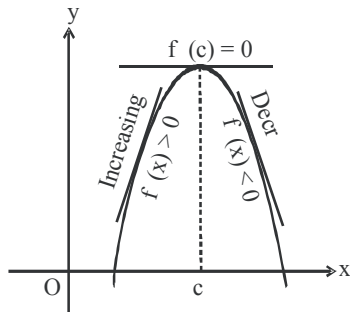
- (a) **Local Maxima** –  $c$  is a point of local maxima if there is an  $h > 0$ , such that  $f(c) \geq f(x)$  for all  $x \in [c-h, c+h]$

The value  $f(c)$  is called local maximum value of  $f$ .

- (b) **Local Minima** –  $c$  is a point of local minima if there is an  $h > 0$ , such that  $f(c) \leq f(x)$  for all  $x \in [c-h, c+h]$

The value of  $f(c)$  is known as the local minimum value of  $f$ .

**Geometrically** – If  $x = c$  is a point of local maxima of  $f$ , then



$f$  is increasing (i.e.,  $f'(x) > 0$ ) in the interval  $(c-h, c)$  and decreasing (i.e.,  $f'(x) < 0$ ) in the interval  $(c, c+h)$

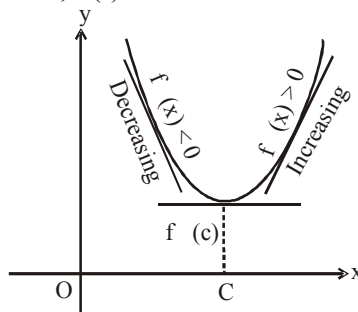
$$\Rightarrow f'(c) = 0$$

Similarly, if  $x = c$  is a point of local minima of  $f$ , then  $f$  is decreasing (i.e.,  $f'(x) < 0$ ) in the interval  $(c-h, c)$  and increasing (i.e.,  $f'(x) > 0$ ) in the interval  $(c, c+h)$ .

$$\Rightarrow f'(c) = 0$$

#### 10. Test of Local Maxima and Minima –

- (i) Let  $f$  be a differentiable function defined on an open interval  $I$  and  $c \in I$  be any point.  $f$  has a local maxima or a local minima at  $x = c$ ,  $f'(c) = 0$



- (ii) If  $f'(x)$  changes sign from positive to negative as  $x$  increases from left to right through  $c$  i.e.,  
 (a)  $f'(x) > 0$  at every point in  $(c-h, c)$   
 (b)  $f'(x) < 0$  at every point in  $(c, c+h)$   
 Then  $c$  is called a point of local maxima of  $f$  and  $f(c)$  is local maximum value of  $f$ .
- (iii) If  $f'(x)$  changes sign from negative to positive as  $x$  increase from left to right through  $c$  i.e.,  
 (a)  $f'(x) < 0$  at every point in  $(c-h, c)$   
 (b)  $f'(x) > 0$  at every point in  $(c, c+h)$   
 Then  $c$  is called a point of local minima of  $f$  and  $f(c)$  is a local minimum value of  $f$ .
- (iv) If  $f'(x)$  does not change sign as  $x$  increases through  $c$ , then  $c$  is neither a point of local maxima nor a point of local minima. Such a point is called point of inflexion.

#### 11. Second Derivative Test of Local Maxima and Minima – let $f$ be a twice differentiable function defined on an interval $I$ and $c \in I$ and $f$ be differentiable at $c \in I$ , then,

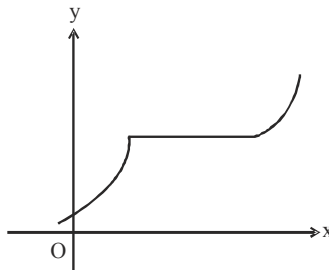
- (i)  $x = c$  is a local maxima,  
 if  $f'(c) = 0$  and  $f''(c) < 0$ .  
 $f(c)$  is the local maximum value of  $f$
- (ii)  $x = c$  is a local minima, if  $f'(c) = 0$  and  $f''(c) > 0$   
 $f(c)$  is the local minimum value of  $f$ .
- (iii) Point of inflexion If  $f'(c) = 0$  and  $f''(c) = 0$   
 Test fails. Then we apply first derivative test and find whether  $c$  is a point of local maxima, local minima or a point of inflexion.

**12. To find absolute maximum value or absolute minimum value –**

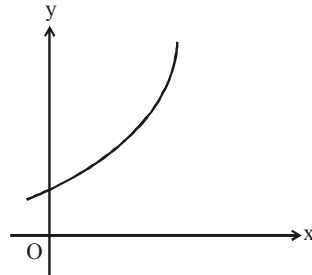
- (i) Find all the critical points where  $f'(x) = 0$
- (ii) Consider the end point also.
- (iii) Calculate the functional values at all the points found in step (i) and (ii)
- (iv) Identify the maximum and minimum values out of the values calculated in step (iii). These are absolute maximum and absolute minimum values.

**CONNECTING CONCEPTS**

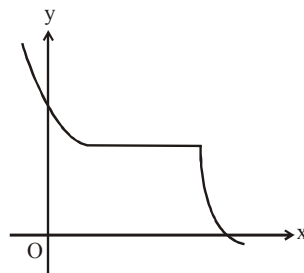
1. **Increasing Function** –  $f$  is said to be increasing on  $I$ , if  $x_1 < x_2$  on  $I$ , then  $f(x_1) \leq f(x_2)$ . for all  $x_1, x_2 \in I$ .



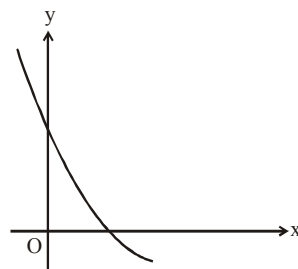
2. **Strictly Increasing function** –  $f$  is said to be strictly increasing on  $I$ , if  $x_1 < x_2$  in  $I$  then  $f(x_1) < f(x_2)$  for all  $x_1, x_2 \in I$ .



3. **Decreasing function** –  $f$  is said to be decreasing function on  $I$ , if  $x_1 < x_2$  in  $I$ , then  $f(x_1) \geq f(x_2)$  for all  $x_1, x_2 \in I$ .



4. **Strictly Decreasing function** –  $f$  is said to be strictly decreasing function on  $I$ , if  $x_1 > x_2$  in  $I$  then  $f(x_1) > f(x_2)$  for all  $x_1, x_2 \in I$ .



**5. Particular case of tangent** – Let  $m = \tan \theta$

If  $\theta = 0$ ,  $m = 0$

Equation of tangent is  $y - y_0 = 0$  i.e.,  $y = y_0$

If  $\theta = \frac{\pi}{2}$ ,  $m$  is not defined.

$$\therefore (x - x_0) = \frac{1}{m} (y - y_0)$$

when  $\theta = \frac{\pi}{2}$ ,  $\cot \frac{\pi}{2} = 0$

$\therefore$  Equation of tangent is  $x - x_0 = 0$  or  $x = x_0$