1 RELATIONS AND FUNCTIONS

KEY CONCEPT INVOLVED

1. **Relations -** Let A and B be two non-empty sets then every subset of $A \times B$ defines a relation from A to B and every relation from A to B is a subset of $A \times B$.
Let $R \subseteq A \times B$ and $(a, b) \in R$. then we say that a is related to b by the relation R as aRb. If $(a, b) \notin R$ as $a \not R\ b$.
2. **Domain and Range of a Relation -** Let R be a relation from A to B, that is, let $R \subset A \times B$. then *Domain R* = {a : a ∈ A, (a, b) ∈ R for some b ∈ B} i.e. dom. R is the set of all the first elements of the ordered pairs which belong to R. *Range R* = (b : b ∈ B, (a, b) ∈ R for some a ∈ A} i.e. range R is the set of all the second elements of the ordered pairs which belong to R. Thus Dom. $R \subset A$, Range $R \subset B$.
3. **Inverse Relation -** Let $R \subset A \times B$ be a relation from A to B. Then inverse relation $R^{-1} \subset B \times A$ is defined by R^{-1} {(b, a) : (a, b) ∈R}
It is clear that
(i) $aRb = bR^{-1} a$
(ii) dom. R^{-1} = range R and range R^{-1} = dom R.
(iii) $(R^{-1})^{-1} = R$.
4. **Composition of Relation -** Let $R \subset A \times B$, $S \subset B \times C$ be two relations. Then composition of the relations R and S is denoted by $SoR \subset A \times C$ and is defined by $(a, c) \in (SoR)$ iff $b \in B$ such that $(a, b) \in R$, $(b, c) \in S$.
5. **Relations in a set -** let A (≠ ϕ) be a set and $R \subset A \times A$ i.e. R is a relation in the set A.
6. **Reflexive Relations -** R is a reflexive relation if $(a, a) \in R$, ∀ a ∈R it should be noted that if for any a ∈A such that $a \not R\ a$. then R is not reflexive.
7. **Symmetric Relation -** R is called symmetric relation on A if $(x, y) \in R \Rightarrow (y, x) \in R$.
i.e. if x is related to y, then y is also related to x.
It should be noted that R is symmetric iff $R^{-1} = R$.
8. **Anti Symmetric Relations -** R is called an anti symmetric relation if (a, b) ∈R and (b, a) ∈R ⇒ a = b.
Thus if a ≠ b then a may be related to b or b may be related to a but never both.
9. **Transitive Relations -** R is called a transitive relation if (a, b) ∈R (b, c) ∈R ⇒ (a, c) ∈R
10. **Identity Relations -** R is an identity relation if (a, b) ∈R iff a = b. i.e. every element of A is related to only itself and always identity relation is reflexive symmetric and transitive.
11. **Equivalence Relations -** a relation R in a set A is called an equivalence relation if
(i) R is reflexive i.e. (a, a) ∈R ∀ a ∈ A
(ii) R is symmetric i.e. (a, b) ∈R ⇒ (b, a) ∈R
(iii) R is transitive i.e. (a, b), (b, c) ∈R ⇒ (a, c) ∈R.
12. **Functions -** Suppose that to each element in a set A there is assigned, by some rule, an unique element of a set B. Such rules are called functions. If we let f denote these rules, then we write $f : A \to B$ as f is a function of A into B.
13. **Equal Functions -** If f and g are functions defined on the same domain A and if f (a) = g (a) for every a ∈A, then f = g.

14. Constant Functions - Let $f : A \to B$. If $f(a) = b$, a constant, for all $a \in A$, then f is called a constant function. Thus f is called a constant function if range f consists of only one element.

15. Identity Functions - A function f is such that $A \to A$ is called an identity function if $f(x) = x, \ \forall \ x \in A$ it is denoted by I_A.

16. One-One Functions (Injective) - Let $f : A \to B$ then f is called a one-one function. If no two different elements in A have the same image i.e. different elements in A have different elements in B.
Denoted by symbol f is one-one if

$$f(a) = f(a') \Rightarrow a = a'$$

i.e. $a \neq a' \Rightarrow f(a) \neq f(a')$

A mapping which is not one-one is called many one function.

17. Onto functions (Surjective) - In the mapping $f : A \to B$, if every member of B appears as the image of atleast one element of A, then we say "f is a function of A onto B or simply f is an onto functions" Thus f is onto iff $f(A) = B$

i.e. range = codomain

A function which is not onto is called into function.

18. Inverse of a function - Let $f : A \to B$ and $b \in B$ then the inverse of b i.e. $f^{-1}(b)$ consists of those elements in A which are mapped onto b i.e. $f^{-1}(b) = \{x ; x \in A, f(x) \in b\}$

$\therefore \ f^{-1}(b) \subset A$, $f^{-1}(b)$ may be a null set or a singleton.

19. Inverse Functions - Let $f : A \to B$ be a one-one onto-function from A onto B. Then for each $b \in B$. $f^{-1}(b) \in A$ and is unique. So, $f^{-1} : B \to A$ is a function defined by $f^{-1}(b) = a$, iff $f(a) = b$.

Then f^{-1} is called the inverse function of f. If f has inverse function, f is also called invertible or non-singular.

Thus f is invertible (non-singular) iff it is one-one onto (bijective) function.

20. Composition Functions - Let $f : A \to B$ and $g : B \to C$, be two functions,
Then composition of f and g denoted by $gof : A \to C$ is defined by $(gof)(a) = g\{f(a)\}$.

21. Binary Operation - A binary operation $*$ on a set A is a function $* : A \times A \to A$. We denote $*(a, b)$ by $a * b$

22. Commutative Binary Operation - A binary operation $*$ on the set A is commutative if for every $a, b \in A$, $a * b = b * a$.

23. Associative Binary Operation - A binary operation $*$ on the set A is associative if $(a * b) * c = a * (b * c)$.

24. An Identity Element e for Binary Operation - Let $* : A \times A \to A$ be a binary operation. There exists an element $e \in A$ such that $a * e = a = e * a \ \forall \ a \in A$, then e is called an identity element for Binary Operation $*$.

25. Inverse of an Element a - Let $* : A \times A \to A$ be a binary operation with identity element e in A. an element $a \in A$ is invertible w.r.t. binary operation $*$, if there exists an element b in A such that $a * b = e = b * a$. and b is called the inverse of a and is denoted by a^{-1}.

CONNECTING CONCEPTS

1. In general $gof \neq fog$.

2. $f : A \to B$, be one-one, onto then
$f^{-1} of = I_A$ and $fof^{-1} = I_B$

3. $f : A \to B, g : B \to C, h : C \to D$
then $(hog) of = ho(gof)$.

4. $f : A \to B, g : B \to C$ be one-one and onto then $gof : A \to C$ is also one-one onto and $(gof)^{-1} = f^{-1} o \ g^{-1}$.

5. Let : $A \to B$, then $I_B of = f$ and $foI_A = f$. It should be noted that foI_B is not defined since for $(foI_B)(x) = fo\{I_B(x)\} = f(x)$
$I_B(x)$ exist when $x \in B$ and $f(x)$ exist when $x \in A$

6. $f : A \to B, g : B \to C$ are both one-one, then $gof : A \to C$ is also one-one it should be noted that for gof to be one-one f must be one-one.

7. If $f : A \to B$ $g : B \to C$ are both onto then gof must be onto. However, the converse is not true. But for gof to be onto g must be onto.

8. The domain of the functions

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)\, g(x)$$

is given by (dom. f) $\cap$ (dom g) while domain of the function $(f/g)(x) = \frac{f(x)}{g(x)}$ is given by.

(dom f) $\cap$ (dom. g) – $\{x : g(x) = 0\}$

9. If O (A) = m, O (B) = n, then total number of mappings from A to B is n^m.

10. If A and B are finite sets and O (A) = m, O (B) = n, $m \le n$.

Then number of injection (one-one) from A to B is ${}^nP_m = \frac{n!}{(n-m)!}$

11. If $f : A \to B$ is injective (one-one), then $O(A) \le O(B)$.

12. If $f : A \to B$ is surjective (onto), then $O(A) \ge O(B)$.

13. If $f : A \to B$ is bijective (one-one onto), then O (A) = O (B).

14. Let $f : A \to B$ and O (A) = O (B), then f is one-one $\Leftrightarrow$ it is onto.

15. Let $f : A \to B$ and $X_1, X_2 \subseteq A$, then f is one-one iff $f(X_1 \cap X_2) = f(X_1) \cap f(X_2)$

16. Let $f : A \to B$ and $X \subseteq A$, $Y \subseteq B$, then in general $f^{-1}(f(x)) \subseteq X$, $f(f^{-1}(y)) \subseteq Y$

If f is one-one onto $f^{-1}(f(x)) = x$, $f(f^{-1}(y)) = Y$.