10 VECTOR ALGEBRA

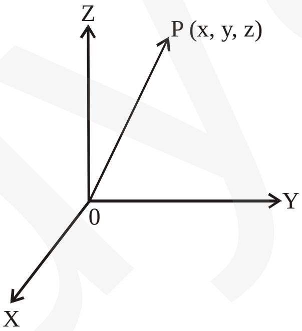
KEY CONCEPT INVOLVED

1. **Vector –** A vector is a quantity having both magnitude and direction, such as displacement, velocity, force and acceleration.
   AB is a directed line segment. It is a vector $\overrightarrow{AB}$ and its direction is from A to B.

   $A \longrightarrow B$

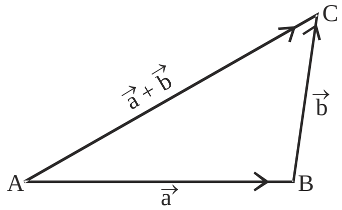
   Initial Points – The point A where from the vector $\overrightarrow{AB}$ starts is known as initial point.
   Terminal Point – The point B, where it ends is said to be the terminal point.
   Magnitude – The distance between initial point and terminal point of a vector is the magnitude or length of the vector $\overrightarrow{AB}$. It is denoted by $|\overrightarrow{AB}|$ or AB.

2. **Position Vector** – Consider a point p (x, y, z) in space. The vector $\overrightarrow{OP}$ with initial point, origin O and terminal point P, is called the position vector of P.



3. **Types of Vectors**
   (i) **Zero Vector Or Null Vector** – A vector whose initial and terminal points coincide is known as zero vector ($\vec{O}$).
   (ii) **Unit Vector** – A vector whose magnitude is unity is said to be unit vector. It is denoted as $\hat{a}$ so that $|\hat{a}| = 1$.
   (iii) **Co-initial Vectors** – Two or more vectors having the same initial point are called co-initialvectors.
   (iv) **Collinear Vectors** – If two or more vectors are parallel to the same line, such vectors are known as collinear vectors.
   (v) **Equal Vectors** – If two vectors $\vec{a}$ and $\vec{b}$ have the same magnitude and direction regardless of the positions of their initial points, such vectors are said to be equal *i.e.*, $\vec{a} = \vec{b}$.
   (vi) **Negative of a vector** – A vector whose magnitude is same as that of a given vector $\overrightarrow{AB}$, but the direction is opposite to that of it, is known as negative of vector $\overrightarrow{AB}$ *i.e.*, $\overrightarrow{BA} = -\overrightarrow{AB}$

4. **Sum of Vectors**
   (i) **Sum of vectors $\vec{a}$ and $\vec{b}$** let the vectors $\vec{a}$ and $\vec{b}$ be so positioned that initial point of one coincides with terminal point of the other. If $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{BC}$. Then the vector $\vec{a} + \vec{b}$ is represented by the third side of ΔABC. *i.e.*, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ ...(i)



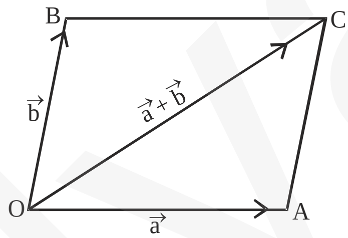
This is known as the triangle law of vector addition.
Further $\overrightarrow{AC} = -\overrightarrow{CA}$

$$\overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{CA} \qquad \therefore \quad \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$$

when sides of a triangle ABC are taken in order i.e. initial and terminal points coincides. Then

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$$

(ii) **Parallelogram law of vector addition** – If the two vectors $\vec{a}$ and $\vec{b}$ are represented by the two adjacent sides OA and OB of a parallelogram OACB, then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal OC of parallelogram through their common point O *i.e.*, $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$

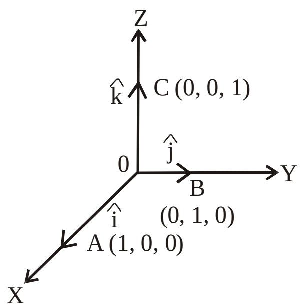


5. Multiplication of Vector by a Scalar – Let $\vec{a}$ be the given vector and λ be a scalar, then product of λ and $\vec{a} = \lambda\vec{a}$

(i) when λ is +ve, then $\vec{a}$ and $\lambda\vec{a}$ are in the same direction.

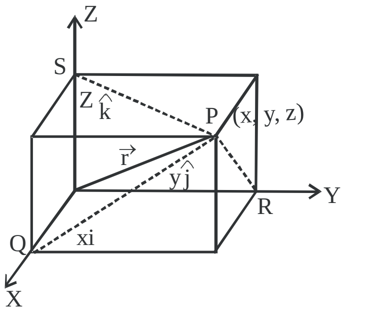
(ii) when λ is –ve. then $\vec{a}$ and $\lambda\vec{a}$ are in the opposite direction. Also $|\lambda\,\vec{a}| = |\lambda|\,|\vec{a}|$.

6. Components of Vector – Let us take the points A (1, 0, 0), B (0, 1, 0) and C (0, 0, 1) on the coordinate axes OX, OY and OZ respectively. Now, $|\overrightarrow{OA}| = 1$, $|\overrightarrow{OB}| = 1$ and $|\overrightarrow{OC}| = 1$, Vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ each having magnitude 1 is known as unit vector. These are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$.



Consider the vector $\overrightarrow{OP}$, where P is the point (x, y, z). Now OQ, OR, OS are the projections of OP on coordinates axes.

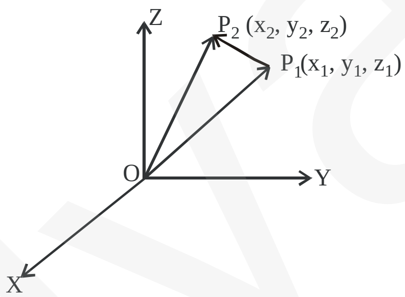
$\therefore \quad OQ = x, OR = y, OS = z \qquad \therefore \quad \overrightarrow{OQ} = x\hat{i}, \quad \overrightarrow{OR} = y\hat{j}, \quad \overrightarrow{OS} = z\hat{k}$



$\Rightarrow \qquad \overrightarrow{OP} = x\hat{i}, + y\hat{j}, + z\hat{k} \quad , \qquad |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2} = |\vec{r}|$

x, y, z are called the scalar components and $x\hat{i}$, $y\hat{j}$, $z\hat{k}$ are called the vector components of vector $\overrightarrow{OP}$.

7. Vector joining two points – Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2\ z_2)$ be the two points. Then vector joining the points P_1 and P_2 is $\overrightarrow{P_1P_2}$. Join P_1, P_2 with O. Now $\overrightarrow{OP_2} = \overrightarrow{OP_1} + \overrightarrow{P_1P_2}$ (by triangle law)



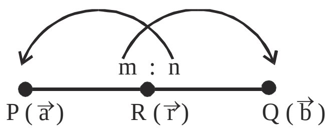
$\therefore \qquad \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$

$= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) = (x_2 - x_1)\ \hat{i} + (y_2 - y_1)\ \hat{j} + (z_2 - z_1)\ \hat{k}$

$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

8. Section Formula

(i) A line segment PQ is divided by a point R in the ratio m : n internally *i.e.*, $\frac{PR}{RQ} = \frac{m}{n}$

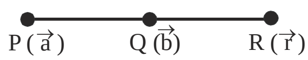


If $\vec{a}$ and $\vec{b}$ are the position vectors of P and Q then the position vector $\vec{r}$ of R is given by

$$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$$

If R be the mid-point of PQ, then $\vec{r} = \frac{\vec{a} + \vec{b}}{2}$

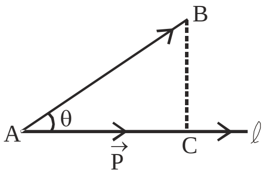
(ii) when R divides PQ externally, i.e., $|\vec{a}|| \cdot \vec{b}|\hat{n}$



Then $\vec{r} = \dfrac{m\vec{b} - n\vec{a}}{m - n}$

9. Projection of vector along a directed line – Let the vector $\overrightarrow{AB}$ makes an angle θ with directed line ℓ.

Projection of AB on ℓ = $|\overrightarrow{AB}| \cos\theta = \overrightarrow{AC} = \vec{p}$.



The vector $\vec{p}$ is called the projection vector. Its magnitudes is $|\vec{b}|$, which is known as projection of vector $\overrightarrow{AB}$. The angle θ between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is given by

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}, \quad \text{Now projection AC} = |\overrightarrow{AB}| \cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AC}|}$$

$$= \overrightarrow{AB} \cdot \left(\frac{\overrightarrow{AC}}{|\overrightarrow{AC}|} \right), \quad \text{If } \overrightarrow{AB} = \vec{a}, \text{ then } \overrightarrow{AC} = \vec{a} \cdot \left(\frac{\vec{p}}{|\vec{p}|} \right) = \vec{a} \cdot \hat{p}$$

Thus, the projection of $\vec{a}$ on $\vec{b}$ = $\vec{a} \cdot \left(\dfrac{\vec{b}}{|\vec{b}|} \right) = \vec{a} \cdot \hat{b}$

10. Scalar Product of Two Vectors (Dot Product) – Scalar Product of two vectors $\vec{a}$ and $\vec{b}$ is defined as

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

Where θ is the angle between $\vec{a}$ and $\vec{b}$ $(0 \le \theta \le \pi)$

(i) when $\theta = 0$, then $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| = ab$ Also $\vec{a} \cdot \vec{a} = |\vec{a}|\,|\vec{a}| = a.a = a^2$

$\therefore \quad \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

(ii) when $\theta = \dfrac{\pi}{2}$, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\dfrac{\pi}{2} = 0$

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

11. Vector Product of two Vectors (Cross Product) – The vector product of two non-zero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$ is defined as

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \cdot \hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$.

Unit vector $\hat{n}$ is perpendicular to both vectors $\vec{a}$ and $\vec{b}$ such that $\vec{a} \cdot \vec{b}$ and $\hat{n}$ form a right handed orthogonal system.

(i) If $\theta = 0$, then $\vec{a} \times \vec{b} = 0$, $\therefore \vec{a} \times \vec{a} = 0$

and $\therefore \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

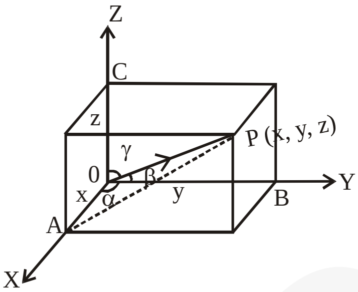
(ii) If $\theta = \Pi/_2$, then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\hat{n}$

$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$

Also, $\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i}$ and $\hat{i} \times \hat{k} = \hat{j}$

CONNECTING CONCEPTS

1. **Direction Cosines –** Let OX, OY, OZ be the positive coordinate axes, P (x, y, z) by any point in the space. Let $\overrightarrow{OP}$ makes angles α, β, γ with coordinate, axes OX, OY, OZ. The angle α, β, γ are known as direction angles, cosine of these angles *i.e.*,



$\cos\alpha, \cos\beta, \cos\gamma$ are called direction cosines of line OP. these direction cosines are denoted by ℓ, m, n *i.e.*, $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

2. **Relation Between, l, m, n and Direction Ratios –**

The perpendiculars PA, PB, PC are drawn on coordinate axes OX, OY, OZ reprectively. Let $|\overrightarrow{OP}| = r$

In ΔOAP, $\angle A = 90°$, $\cos\alpha = \frac{x}{r} = \ell$, $\therefore$ $x = \ell r$, In ΔOBP. $\angle B = 90°$, $\cos\beta = \frac{y}{r} = m$ $\therefore$ $y = mr$

In ΔOCP, $\angle C = 90°$, $\cos\gamma = \frac{z}{r} = n$, $\therefore$ $z = nr$

Thus the coordinates of P may b expressed as (ℓr, mr, nr)

Also, $OP^2 = x^2 + y^2 + z^2$, $r^2 = (lr)^2 + (mr)^2 + (nr)^2$ $\Rightarrow$ $\ell^2 + m^2 + n^2 = 1$

Set of any there numbers, which are proportional to direction cosines are called direction ratio of the vactor. Direction ratio are denoted by a, b and c.

The numbers ℓr mr and nr, proportional to the direction cosines, hence, they are also direction ratios of vector $\overrightarrow{OP}$.

3. **Properties of Vector Addition –**

   1. For two vectors $\vec{a}, \vec{b}$ the sum is commutative *i.e.*, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
   2. For three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, the sum of vectors is associative i.e.,

      $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

4. **Additive Inverse of Vector $\vec{a}$** – If there exists vector – $\vec{a}$ such that $\vec{a} + (-\vec{a}) = \vec{a} - \vec{a} = \vec{0}$ then – $\vec{a}$ is called the additure inverse of $\vec{a}$

5. **Some Properties –** Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

   (i) $\vec{a} + \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

   (ii) $\vec{a} = \vec{b}$ or $(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$ $\Rightarrow$ $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$

   (iii) $\lambda\vec{a} = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

   (iv) $\vec{a}$ and $\vec{b}$ are parallel, if and only if there exists a non zero scalar λ such that $\vec{b} = \lambda\vec{a}$

i.e., $b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda\,(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

$\therefore\ b_1 = \lambda a_1,\ b_2 = \lambda a_2,\ b_3 = \lambda a_3 \quad \therefore\ \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$

6. Properties of scalar product of two vectors (Dot Product)

(i) $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then, $\vec{a}\cdot\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$, $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$ $\quad \therefore\ \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\cdot\sqrt{b_1^2 + b_2^2 + b_3^2}}$

(ii) $\vec{a}\cdot\vec{b}$ is commutative *i.e.,* $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

(iii) If α is a scalar, then $(\alpha\vec{a})\cdot\vec{b} = \alpha(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\alpha\vec{b})$

7. Properties of Vector Product of two Vectors (Cross Product) –

(i) (a) If $\vec{a} = 0$ or $\vec{b} = 0$, then $\vec{a}\times\vec{b} = 0$

(b) If $\vec{a} \parallel \vec{b}$, then $\vec{a}\times\vec{b} = 0$

(ii) $\vec{a}\times\vec{b}$ is not commutative

i.e. $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$, but $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

(iii) If $\vec{a}$ and $\vec{b}$ represent adjacent sides of a parallelogram, then its area $|\vec{a}\times\vec{b}|$

(iv) If $\vec{a}, \vec{b}$ represent the adjacent sides of a triangle, then its area = $\frac{1}{2}|\vec{a}\times\vec{b}|$

(v) Distributive property $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

(a) If α be a scalar, then $\alpha\,(\vec{a}\times\vec{b}) = (\alpha\vec{a})\times\vec{b} = \vec{a}\times(\alpha\vec{b})$

(b) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

8. If $\alpha_1\ \beta_1\ \gamma$ are the direction angles of the vector $\vec{a} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$. Then direction cosines of $\vec{a}$ are given as

$\cos\alpha = \frac{a_1}{|\vec{a}|}$, $\cos\beta = \frac{a_2}{|\vec{a}|}$, $\cos\gamma = \frac{a_3}{|\vec{a}|}$

9. **Scalar Product of Two Vectors (Dot Product)** – Scalar Product of two vectors $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ $\left(0 \le \theta < \frac{\pi}{2}\right)$

(i) When $\theta = 0$, then $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|$. Also $\vec{a} \cdot \vec{a}$ $a \cdot a = a^2$

$\therefore \quad \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

(ii) When $\theta = \frac{\pi}{2}$, $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\frac{\pi}{2} = 0$