4 DETERMINANTS

KEY CONCEPTS INVOLVED

1. **Determinant -** (i) A determinant is a particular type of expression written in a special concise form of rows and columns, equal in number.

For example $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is a determinant having 2 rows and 2 columns, hence it is of second order. The numbers a_1, b_1, a_2 , b_2 are called the elements of the determinant. The value of the above determinant of third order is written as $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. It has three rows and three columns.

The number of elements = $3^2 = 9$. In general, the number of elements in a determinant of order n = n^2.

(ii) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is a matrix then determinant of matrix A is written as |A| or det (A) = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

(iii) Only square matrices have determinants.

2. **Minors -** The determinant obtained by deleting the i-the row and j-th column passing through the element a_{ij} is called the minor of element a_{ij} and is denoted by M_{ij}.
3. **Cofactors -** The cofactor of element a_{ij} is $(-1)^{1+j}$ times the determinant obtained by deleting the i-th row and jth column passed through a_{ij} and is denoted by C_{ij} i.e. $C_{ij} = (-1)^{1+j} M_{ij}$
4. **Values of the determinant -** The sum of the products of elements of any row (column) by the corresponding co-factors is equal to the value of the determinant.

let $\Delta = \begin{vmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, Then $\Delta = a_{11}\, c_{11} + a_{12}\, c_{12} + a_{13}\, c_{13}$

5. **Area of a Triangle -** The area of the triangle whose vertices are (x_1 , y_1) , (x_2 , y_2) and (x_3 , y_3) is

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(i) The area is positive, take only absolute value.
(ii) If the three points are collinear, the area of triangle is zero.

6. |AB| = |A| |B|
7. A square matrix is invertible if and only if A is non-singular.
8. **Linear system of Equations -**

Consistent System - The system of equation is said to be consistent if it has one or more then one solutions.

Inconsistent System - The system of equation is inconsistent if it has no solution

Consider the system of equation

$$a_1x + b_1y_+ c_1z = d_1$$
$$a_2x + b_2y_+ c_2z = d_2$$
$$a_3x + b_3y_+ c_3z = d_3$$

let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

The given system of equation can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or $AX = B$

$\therefore$ $X = A^{-1} B$.

9. **Consistence/Inconsistence of system of Equations**

(a) For a non-homogeneous system of equation $AX \neq 0$

(i) If $|A| \neq 0$, $AX = B$ has a unique solution.

(ii) If $|A| = 0$, and $(\text{adj } A)\, B \neq 0$ then the system of equation is inconsistent.

(iii) If $|A| = 0$ and $(\text{adj } A)\, B = 0$, then the system of equation has infinitely many solutions.

(b) For the homogeneous system of equation $AX = 0$

(i) If $|A| \neq 0$, the solution is $x = 0, y = 0, z = 0$. This is called the trivial solution.

(ii) If $|A| = 0$, the system has infinitely many solution. In such a case, we put one of the variables equal to k. let $z = k$, then we find the value of x and y in terms of k.

10. **Adjoint of a Determinant -** The adjoint of a square matrix is the transpose of matrix cofactors. If A_{ij} is the cofactor of a_{ij} of det A or $|a_{ij}|$, the

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

11. **Inverse of a matrix -** Inverse of a matrix A, $A^{-1} = \frac{1}{|A|}$ adj A ; if $|A| \neq 0$ i.e., matrix A is invertible or non-singular.

12. If A is a square matrix, then $A (\text{adj } A) = (\text{adj } A) A = |A| . I$

13. (i) $(AB)^{-1} = B^{-1} . A^{-1}$ (ii) $A^{-1} = (A^{-1})^T$ (iii) $(A^{-1})^{-1} = A$

CONNECTING CONCEPTS

1. The value of the determinant does not change when rows and columns are interchanged. The determinant obtained by interchanging the rows and columns is called the transpose of the determinant and is denoted by Δ^T. Thus $\Delta = \Delta^T$.
2. If all the elements of a row (column) are zero, then the value of the determinant is zero.
3. The interchange of any two rows of the determinant changes its sign.

Thus if Δ^* is the new determinant obtained on interchanging any two rows (columns), then

$$\Delta = -\Delta^*$$

If i-th and j-th row are interchanged then this operation is denoted by $R_i \longleftrightarrow R_j$.

4. If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant. Thus if we apply $R_i \to pR_i$, i.e, each element of i-th row is multiplied by p, then we get

$$\Delta^* = p\Delta \quad \text{or} \quad \Delta = \frac{1}{p}\Delta^* \ (P \neq 0)$$

5. If all the elements of a row (column) are proportional (identical) to the elements of some other row (column) then determinant is zero.

6. If each element of any row (column) is sum of two numbers, the determinant can be expressed as the sum of two determinants of the same order eg.

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

7. The value of a determinant remains unaltered under an operation of the form

$R_i \to R_i + pR_j \to$ similarly, for columns i.e., operation of the form $C_i \to C_i + pC_j + q\,C_k$; $j, k \neq i$

8. If a determinant Δ (x) becomes zero on putting $x = \alpha$, then $(x - \alpha)$ is a factor of Δ (x).

9. Determinant which have all elements equal to zero except the diagonal elements, is equal to the product of the diagonal elements.

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$