5 CONTINUITY AND DIFFERENTIABILITY

KEY CONCEPT INVOLVED

1. **Continuity -** A real valued function f (x) of variable x defined on an interval I is said to be continuous at $x = a \in I$, $\lim_{x \to a} f(x)$ exists, is finite and is equal to f (a).

   $\therefore \lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a-h) = f(a)$, where 'h' is a very small +ve quantity.
2. A function f (x) is said to be continuous in an interval I, if it is continuous at each point of the interval.
3. **Discontinuity -** A function said to be discontinuous at a point x = a, if it is not continuous at this point. This point x = a where the function is not continuous is called the point of discontinuity.
4. Suppose f and g be two real functions continuous at a real number c, then
   (i) $f + g$ is continuous at $x = c$
   (ii) $f - g$ is continuous at $x = c$
   (iii) $f \cdot g$ is continuous at $x = c$
   (iv) $\frac{f}{g}$ is continuous at $x = c$, (provided $g(c) \neq 0$)
5. (i) If g is a continuous function, then $\frac{1}{g}$ is also continuous.
   (ii) Suppose f and g are real valued functions such that (fog) is defined at c. If f and g is continuous at c then (fog) is also continuous at c.
6. **Differentiability -** The concept of differentiability has been introduced in the lower class let f be a real function and c is a point in its domain. The derivative $f'(c)$ of f at c is defined as $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$, provided limit exists

   Thus, $f'(c) = \frac{d}{dx}[f(x)]_c$. $f'(x)$ is defined as $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

   Every differentiable function is continuous.
7. **Algebra of Derivatives -** Let u, v be the function of x.
   (i) $(u \pm v)' = u' \pm v'$
   (ii) $(uv)' = u'v + uv'$
   (iii) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, where $v \neq 0$.
8. **Chain Rule -** If f and g are differentiable functions in their domain, then fog (x) or f g (x) is also differentiable and $(fog)'(x) = f'g(x) \times g'(x)$

   More easily if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

   If y is a function of u, u is a function of v and v is a function of x then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$.
9. **Implicit functions -** An equation in the form $f(x, y) = 0$ in which y is not expressible in terms of x is called as an implicit function of x and y.

Both sides of equations are differentiated term wise with respect to x then from this equation $\frac{dy}{dx}$ is obtained. It may be noted that when a function of y occurs, then differentiate it w.r.t. y and multiply it by $\frac{dy}{dx}$.

Collect the terms containing $\frac{dy}{dx}$ at one side and find $\frac{dy}{dx}$

10. Exponential function - The exponential function with positive base b > 1, is the function $y = b^x$.

(i) The graph of $y = 10^x$ is

(ii) Domain = R

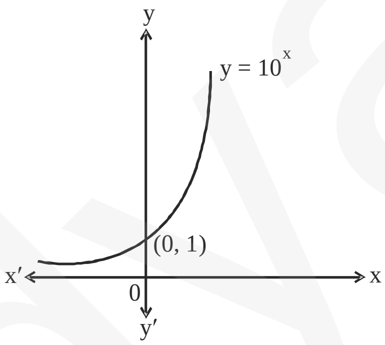
(iii) Range = R^+

(iv) The point (0, 1) always lies on the graph.

(v) It is an increasing function

(vi) As $x \to -\infty$ $y \to 0$

(vii) $\frac{d}{dx} a^x = a^x \log_e a$, $\frac{d}{dx} e^x = e^x$.



11. Logarithmic function - Let b > 1 be a real number. $b^x = a$ may be written as $\log_b a = x$.

(i) The graph of $y = \log_{10} x$ is

(ii) Domain = R^+

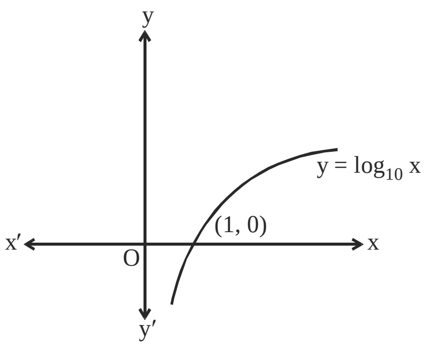
(iii) Range = R

(iv) It is an increasing function.

(v) As $x \to 0$, $y \to -\infty$.

(vi) The function $y = e^x$ and $y = \log_e x$ are the mirror images of each other

(vii) $\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$, $\frac{d}{dx}\log_e x = \frac{1}{x}$



12. Derivatives of functions in Parametric form - The set of equations x = f (t), y = g (t) is called the parametric form of an equation.

Now, $\dfrac{dx}{dt} = f'(t), \dfrac{dy}{dt} = g'(t)$, $\therefore \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ or $\dfrac{g'(t)}{f'(t)}$

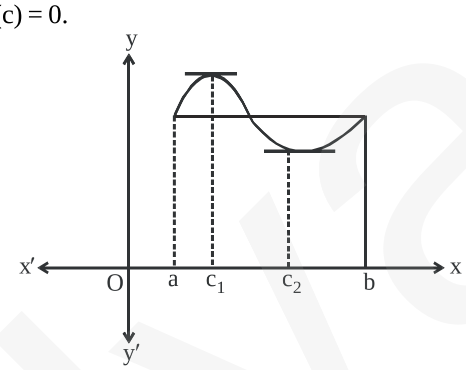
13. **Second order derivative-** let $y = f(x)$ then $\dfrac{dy}{dx} = f'(x)$

If $f'(x)$ is differentiable, then it is again differentiated and get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) \text{ or } \frac{d^2y}{dx^2} = f''(x)$$

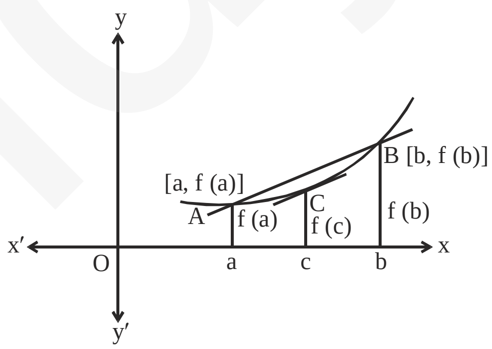
$\dfrac{d^2y}{dx^2}$ or $f''(x)$ is called the second derivative of y or f (x) with respect to x.

14. **Rolle's Theorem -** Let $f : [a, b] \to R$ be continuous an closed interval [a, b] and differentiable an open interval (a, b) such that f (a) = f (b) where a, b are real numbers, then there must exists at least one value $c \in (a, b)$ of x,such that $f'(c) = 0$.



We observe that f (a) = f (b), There exists two point c_1 and $c_2 \in (a, b)$ such that $f'(c_1) = 0$ and $f'(c_2) = 0$, i.e. Tangent at c_1 and c_2 are parallel to x-axis.

15. **Mean Value Theorem-** Let $f : [a, b] \to R$ be a continuous function on the closed interval [a, b] and differentiable in the open interval (a, b), then there must exists at least one value $c \in (a, b)$ of x, such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



Here, $\dfrac{f(b) - f(a)}{b - a}$ is the slope of secant drawn between A [a, f (a)] and B [b, f (b)]. There is at least one point $c \in (a, b)$ of x where slope of the tangent at x = c is parallel to chord AB.

CONNECTING CONCEPTS

Some common type functions as constant function, Identity function, implicit function, Modulus function, Exponential function, and logarithmic function are continuous in their domains.

1. Every polynomial function is differentiable at each $x \in R$.

2. The exponential function a^x, $a > 0$, is differentiable at each $x \in R$

3. Every constant function is differentiable at each $x \in R$.
4. The logarithmic function is differentiable at each point in its demain.
5. Trigonometric and inverse-trigenometric functions are differentiable in their domains.
6. The sum, difference, product and quotient of two differentiable functions is differentiable
7. The composition of differentiable function is differentiable function.
8. (i) $\log_b pq = \log_b p + \log_b q$

(ii) $\log_b \frac{p}{q} = \log_b p - \log_b q$

(iii) $\log_b p^x = x \log_b p$

(iv) $\log_a p = \frac{\log_b p}{\log_b a}$

9. Derivativs of Inverse Trigonometric Functions.

Functions	**Domain**	**Derivative**
$\sin^{-1} x$	$[-1, 1]$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$[-1,1]$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	R	$\frac{1}{1+x^2}$
$\cot^{-1} x$	R	$\frac{-1}{1+x^2}$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\frac{1}{x\sqrt{x^2-1}}$
$\text{cosec}^{-1} x$	$(-\infty, -1) \cup [1, \infty)$	$\frac{-1}{x\sqrt{x^2-1}}$