6 APPLICATION OF DERIVATIVES

KEY CONCEPTS INVOLVED

1. **Rate of change of Quantities –** Let y = f (x) be a function. If the change in one quantity y varies with another quantity x, then $\frac{dy}{dx}$ or $f'(x)$ denotes the rate of change of y with respect to x. $\left.\frac{dy}{dx}\right]_{x = x_0}$ or $f'(x_0)$ represents the rate of change of y w.r.t. x at $x = x_0$.

2. **Increasing and Decreasing function at x_0** A function f is said to be

   (*a*) Increasing on an interval (a, b) if $x_1 < x_2$ in (a, b) $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$ Alternatively, if $f'(x) \geq 0$ for each x in (a, b)

   (*b*) Decreasing on (a, b) if $x_1 < x_2$ in (a, b) $\Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in (a, b)$ Alternatively, if $f'(x) \leq 0$ for each x in (a, b)

3. **Test : Increasing/decreasing/constant function –** Let f be a continuous on [a, b] and differentiable in an open interval (a, b), then.

   (i) f is increasing on [a, b], if $f'(x) > 0$ for each $x \in (a, b)$

   (ii) f is decreasing on [a, b], if $f'(x) < 0$ for each $x \in (a, b)$

   (iii) f is constant on [a, b], if $f'(x) = 0$ for each $x \in (a, b)$

4. **Tangent to a Curve –** Let y = f (x) be the equation of a curve. The equation of the tangent at (x_0, y_0) is $y - y_0 = m(x - x_0)$, where m = slope of the tangent $= \left.\frac{dy}{dx}\right]_{(x_0, y_0)}$ or $f'(x_0)$

5. **Normal to the Curve –** Let y = f (x) be the equation of the curve Equation of the normal at (x_0, y_0) is

$$y - y_0 = -\frac{1}{m}(x - x_0)$$

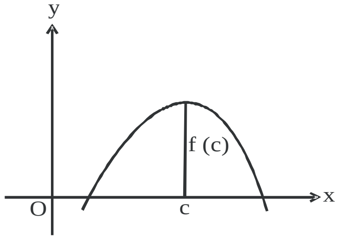
   where m = Slope of the tangent at (x_0, y_0)

$$= \left.\frac{dy}{dx}\right]_{(x_0, y_0)} \text{ or } f'(x_0)$$

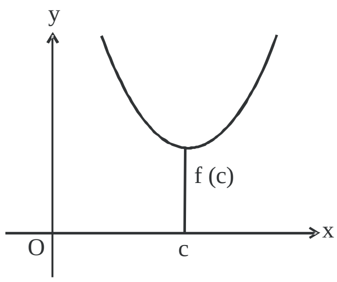
6. **Approximation –** Let y = f (x), Δx be a small increament in x and Δy be the increament in y corrseponding to the increament in x, i.e., $\Delta y = f(x + \Delta x) - f(x)$. Then approximate value of $\Delta y = \left(\frac{dy}{dx}\right)\Delta x$

7. **Maximum Value, Minimum value, Extreme Value –** Let f be a function defined in the interval I, then

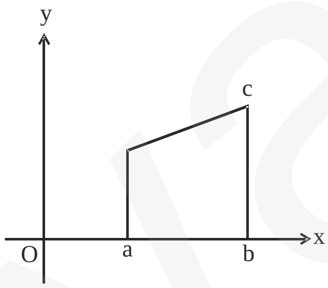
   (i) **Maximum Value –** If there exists a point c in I such that $f(c) \geq f(x)$, for all $x \in I$ then f (c) is called maximum value of f in I. The point c is known as a point of maximum value of f in I.



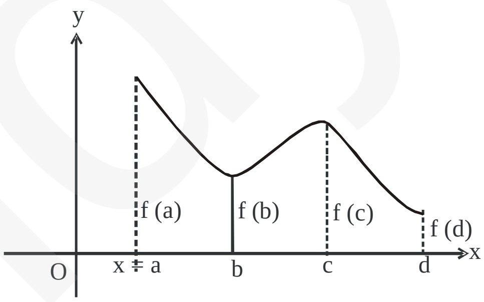
(ii) **Minimum Value –** If three exists a point c in I such that f (c) ≤ f (x), $\forall$ x ∈ I, then f (x) is called the minimum value of f in I. The point c is called as a point of minimum value of f in I



(iii) **Extreme Value –** If there exists a point c in I such that f (c) is either a maximum value or a minimum value of f in I, then f (c) is the extreme value of f (x) in I.
The point c is said to be an extreme point.



8. Absolute Maxima and Minima – let f be a continuous function on an interval I = [a, b]. Then f has the absolute maximum value and f attains it at least once in I. Similarly, f has the absolute minimum value and attains at least once in I

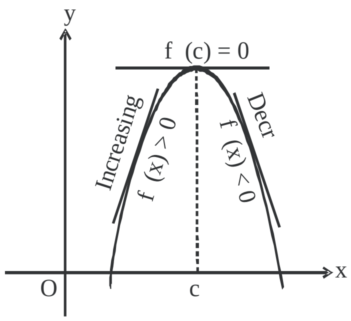


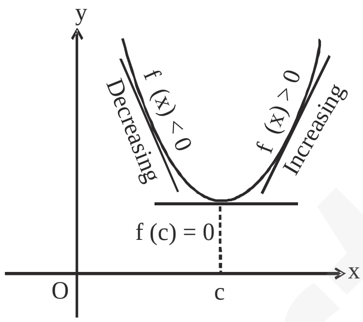
At x = b , there is a local minima
At x = c, there is a local maxima
At x = a, f (a) is the greatest value or absolute max. value.
At x = d, f (d) is the least value or absolute min. value.

9. Local Maxima and Minima – let f be a real valued function and c be an interior point in the domain of f, then

(a) **Local Maxima –** c is a point of local maxima if there is an h > 0, such that f (c) ≥ f (x) for all x ∈ [c – h, c + h)
The value f (c) is called local maximum value of f.

(b) **Local Minima –** c is a point of local minima if there is an h > 0, such that f (c) ≤ f (x) for all x ∈ (c – h c + h)
The value of f (c) is known as the local minimum value of f.
Geometrically – If x = c is a point of local maxima of f, then





f is increasing (i.e., $f'(x) > 0$) in the interval $(c - h, c)$ and decreasing (i.e., $f'(x) < 0$) in the interval $(c, c + h)$

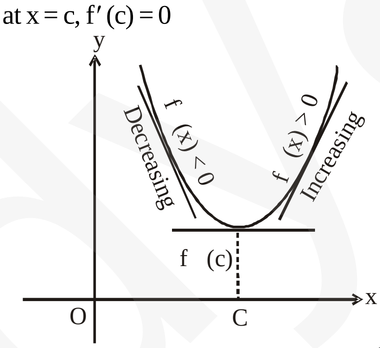
$\Rightarrow \quad f'(c) = 0$

Similarly, if $x = c$ is a point of local minima of f, then f is decreasing (i.e., $f'(x) < 0$) in the interval $(c - h, c)$ and increasing (i.e., $f'(x) > 0$) in the interval $(c, c + h)$.

$\Rightarrow \quad f'(c) = 0$

10. Test of Local Maxima and Minima –

(i) Let f be a differentiable function defined on an open interval I and $c \in I$ be any point. f has a local maxima or a local minima at $x = c$, $f'(c) = 0$



(ii) If $f'(x)$ changes sign from positive to negative as x increases from left to right through c i.e.,
(a) $f'(x) > 0$ at every point in $(c - h, c)$
(b) $f'(x) < 0$ at every point in $(c, c + h)$
Then c is called a point of local maxima of f and f (c) is local maximum value of f.

(iii) If $f'(x)$ changes sign from negative to positive as x increase from left to right through c i.e.,
(a) $f'(x) < 0$ at every point in $(c - h, c)$
(b) $f'(x) > 0$ at every point in $(c, c + h)$
Then c is called a point of local minima of f and f (c) is a local minimum value of f.

(iv) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Such a point is called point of inflection.

11. Second Derivative Test of Local Maxima and Minima – let f be a twice differentiable function defined on an interval I and $c \in I$ and f be differentiable at $c \in I$, then,

(i) $x = c$ is a local maxima,
if $f'(c) = 0$ and $f''(c) < 0$.
f (c) is the local maximum value of f

(ii) $x = c$ is a local minima, if $f'(c) = 0$ and $f''(c) > 0$
f (c) is the local minimum value of f.

(iii) Point of inflection If $f'(c) = 0$ and $f''(c) = 0$
Test fails. Then we apply first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

12. **To find absolute maximum value or absolute minimum value –**

(i) Find all the critical points where $f'(x) = 0$

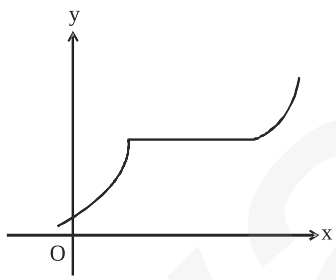
(ii) Consider the end point also.

(iii) Calculate the functional values at all the points found in step (i) and (ii)

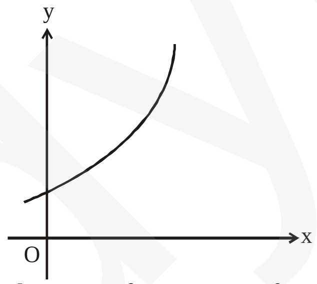
(iv) Identify the maximum and minimum values out of the values calculated in step (iii). These are absolute maximum and absolute minimum values.

CONNECTING CONCEPTS

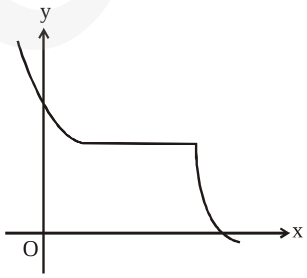
1. **Increasing Function –** f is said to be increasing on I, if $x_1 < x_2$ on I, then $f(x_1) \leq f(x_2)$. for all $x_1, x_2 \in I$.



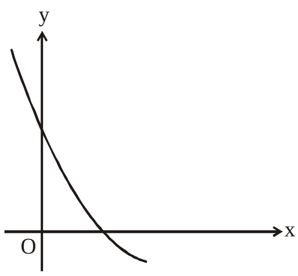
2. **Strictly Increasing function –** f is said to be strictly increasing on I, if $x_1 < x_2$ in I then $f(x_1) < f(x_2)$ for all $x_1, x_2 \in I$.



3. **Decreasing function –** f is said to be decreasing function on I, if $x_1 < x_2$ in I, then $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in I$.



4. **Strictly Decreasing function –** f is said to be strictly decreasing function on I, if $x_1 > x_2$ in I then $f(x_1) > f(x_2)$ for all $x_1, x_2 \in I$.



5. Particular case of tangent – Let $m = \tan\theta$

If $\theta = 0, m = 0$

Equation of tangent is $y - y_0 = 0$ i.e., $y = y_0$

If $\theta = \frac{\pi}{2}$, m is not defined.

$\therefore \quad (x - x_0) = \frac{1}{m}(y - y_0)$

when $\theta = \frac{\pi}{2}, \cot\frac{\pi}{2} = 0$

$\therefore$ Equation of tangent is $x - x_0 = 0$ or $x = x_0$