7 INTEGRALS

KEY CONCEPTS INVOLVED

1. **Integration –** The process of finding the function f (x) whose differential coeffiicient w.r.t. 'x', denoted by F (x) is given, is called the integration of f (x) w.r.t. x and is written as $\int F(x)\,dx = f(x)$
Thus, integration is an inverse process of differentiation or integration is anti of differentiation.
The differential coefficient of a constant is zero. Thus if c is an arbitrary constant independent of x. then
$\frac{d}{dx}[f(x) + c] = F(x)$ Thus $\int F(x)\,dx = f(x) + c$
The arbitrary constant c is called the constant of integration.

2. **Integration by Substitution**

(a) **To evaluate the integral $\int f(ax + b)\,dx$**

Put ax + b = t, so that adx = dt *i.e.*, $dx = \frac{1}{a}dt$

$\int f(ax+b)\,dx = \int f(t)\cdot\frac{1}{a}dt = \frac{1}{a}F(t)$, where $\int f(t)\,dt = F(t) = F(ax+b)$

If a function is not in some suitable form to find the integration, then we transform it into some suitable form by changing the independent variable x to t by substituting x = g (t).

Consider $I = \int f(x)\,dx$

Put $x = g(t)$, so that $\frac{dx}{dt} = g'(t)$

We write $dx = g'(t)\,dt$

Thus $I = \int f(x)\cdot dx = \int f(g(t))\,g'(t)\,dt$

But it is very important to guess, what will be the useful substitution.

(b) $\int \frac{f'(x)}{f(x)}dx = \log f(x) + c$

(c) $\int [f(x)]^n f'(x)\,dx = f(x)^{n+1}/(n+1) + c$

(d) Some important substitutions

function	Substitutions
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

3. **Trigonometrical transformations** – For the integration of the trigonometrical products such as $\sin^2 x$, $\cos^2 x$, $\sin^3 x$, $\cos^3 x$, sin ax cos bx etc.they are expressed as the sum or difference of the sines and cosines of multiples of angles.

4. **Integration of Some Special Integrals –**

(a) For $\int \frac{dx}{ax^2 + bx + c}$, $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ and $\int \sqrt{ax^2 + bx + c}\, dx$

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

Put $x + \frac{b}{2a} = t$, $\therefore$ $dx = dt$, $\frac{4ac - b^2}{4a^2} = \pm k^2$, $ax^2 + bx + c$ changes to $t^2 + k^2$, $t^2 - k^2$ or $k^2 - t^2$

(b) For $\int \frac{(px + q)\, dx}{ax^2 + bx + c}$, $\int \frac{(px + q)\, dx}{\sqrt{ax^2 + bx + c}}$, $\int (px + q)\sqrt{(ax^2 + bx + c)}\, dx$

Put $px + q = A \frac{d}{dx}(ax^2 + bx + c) + B$

Compare the two sides and find the value of A and B.

$$\text{Thus } \int \frac{px + q}{ax^2 + bx + c}\, dx = \int \frac{A \frac{d}{dx}(ax^2 + bx + c) + B}{(ax^2 + bx + c)}$$

$$= A \int \frac{\frac{d}{dx}(ax^2 + bx + c)}{(ax^2 + bx + c)}\, dx + B \int \frac{dx}{(ax^2 + bx + c)}$$

$$\text{Similarly} \int \frac{px + q}{\sqrt{ax^2 + bx + c}}\, dx = A \int \frac{\frac{d}{dx}(ax^2 + bx + c)}{\sqrt{ax^2 + bx + c}}\, dx + B \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

same as do $\int (px + q)\sqrt{ax^2 + bx + c}\, dx$.

(c) For $\int \frac{dx}{(x + k)\sqrt{ax^2 + bx + c}}$ put $x + k = \frac{1}{t}$

(d) For $\int \frac{dx}{\sqrt{(x - \alpha)(x - \beta)}}$, $\int \sqrt{\frac{x - \alpha}{\beta - x}}\, dx$

$\int \sqrt{(x - \alpha)(x - \beta)}\, dx$, Put $x = \alpha \cos^2\theta + \beta \sin^2\theta$

(e) For $\int \frac{dx}{a + b\cos x}$, $\int \frac{dx}{a + b\sin x}$, $\int \frac{dx}{a + b\cos x + c\sin x}$

$\sin x = \left(2\tan\frac{x}{2}\right) \Big/ \left(1 + \tan^2 x/2\right)$, $\cos x = \left(1 - \tan\frac{x}{2}\right) \Big/ \left(1 + \tan^2 x/2\right)$ then put $\tan x/2 = t$

(f) For $\int \frac{p\cos x + q\sin x}{a + b\cos x + b\sin x}\, dx$

Put $p\cos x + q\sin x = A(a + b\cos x + b\sin x) + B$ differential of $(a + b\cos x + b\sin x) + C$

A, B and C can be calculated by equating the coefficients of cos x. sin x and the constant terms.

5. **Integration by parts** $\int u \cdot v\, dx = u \cdot \int v\, dx - \int \left[\frac{du}{dx} \cdot \int v\, dx\right] dx$

i.e., the integral of the product of two functions = (first function) × (Integral of the second function – Integral of {(dfferential of first function) x (Integral of second function)}

This formula is called integration by parts.

6. **Partial Integration** – To Evaluate $\int \frac{P(x)}{Q(x)} dx$

The rational functions which we shall consider here for integration purposes will be those whose denominators can be factorised into linear and quadratic factors.

If $\frac{P(x)}{Q(x)}$ is improper fraction, i.e., degree of numerator is equal or greater than the degree of denominator.

Then first we reduce in proper rational function as $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$ where T (x) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is a proper rational function.

After this, the integration can be carried out easily using the already known methods. The following Table 7.1 indicates the types of simpler partial fractions that are to be associated with various kind of rational functions.

Table 7.1

S. No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px - q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px + q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-b)^2}$
3.	$\frac{px^2 + qx + r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2 + qx + r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5.	$\frac{px^2 + qx + r}{(x-a)(x^2+bx+c)}$ Where $x^2 + bx + c$ can not be factorised further	$\frac{A}{x-a} + \frac{Bx+c}{x^2+bx+c}$

In the above table, A, B and C are real numbers to be determined suitably.

7. **Definite Integral** – The definite integral of f(x) between the limits a to b i.e. in the interval [a,b] is denoted by $\int_a^b f(x)\,dx$ and is defined as follows. $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$ where $\int f(x)\,dx = F(x)$

8. **General Properties of Definite Integrals** –

Prop. I $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

Prop. II $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Prop. III $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$

Prop. IV $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

In particualr $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Prop. V $\int_0^{2a} f(x)\,dx$

Prop. V $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx,$ if f (x) is even function

$\int_{-a}^{a} f(x)\,dx = 0,$ if f (x) is odd function

Prop. VI $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Prop. VII $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx,$ if $f(2a-x) = f(x)$

$\int_0^{2a} f(x)\,dx = 0,$ if $f(2a-x) = -f(x)$

9. Definite Integral as the limit of a sum

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + f(a+2h) + \cdots + f < a + (n-1)\,h)]$$

or $$\int_a^b f(x)\,dx = \lim_{h\to 0} h\,[f(a+h) + f(a+2h) + f(a+3h) + \cdots + f(a+nh)]$$

where, $h = \dfrac{b-a}{n}$

$\dfrac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f\{v(x)\}\dfrac{d}{dx}v(x) - f\{u(x)\}\dfrac{d}{dx}u(x)$ this rule is called leibnitz's is Rule.

CONNECTING CONCEPTS

1. Integration is an operation on function

2. $\int [k_1 f_1(x) + k_2 f_2(x) + \ldots\ldots\ldots\ldots + k_n f_n(x)]dx$

$= k_1\int f_1(x)\,dx + k_2\int f_2(x)\,dx + \ldots\ldots\ldots\ldots + k_n\int f_n(x)\,dx$

3. All functions are not integrable and the integral of a function is not unique.

4. If a polynomial function of a degree n is integrated we get a polynomial of degree n + 1

4. **Integration by using standard formulae –**

1. $\int k\,dx = kx + c$, k is constant
2. $\int kf(x)\,dx = k\int f(x)\,dx + c$
3. $\int (f_1(x) \pm f_2(x)]\,dx = \int f_1(x)dx \pm \int f_2(x)\,dx + c$
4. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c\ (n \neq -1)$
5. $\int \dfrac{1}{x}\,dx = \log_e|x| + c$
6. $\int a^x\,dx = \dfrac{a^x}{\log_e a} + c,\ a > 0$
7. $\int e^x\,dx = e^x + c$

5. $\int \sin x \, dx = -\cos x + c$

6. $\int \cos x \, dx = \sin x + c$

7. $\int \sec^2 x \, dx = \tan x + c$

8. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$

9. $\int \sec x \tan x \, dx = \sec x + c$

10. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

11. $\int \tan x \, dx = \log |\sec x| + c = -\log |\cos x| + c$

12. $\int \cot x \, dx = \log |\sin x| + c$

13. $\int \sec x \, dx = \log |\sec x + \tan x| + c$

14. $\int \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| + c$

15. $\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + c \quad \text{or} \quad -\cos^{-1} x + c$

16. $\int \frac{1}{1 + x^2} dx = \tan^{-1} x + c \quad \text{or} \quad -\cot^{-1} x + c$

17. $\int \frac{1}{x\sqrt{x^2 - 1}} dx = \sec^{-1} x + c \quad \text{or} \quad -\operatorname{cosec}^{-1} x + c$

18. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

19. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left|\frac{x - a}{x + a}\right| + c, \ x > a$

20. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left|\frac{a + x}{a - x}\right| + c, \ x < a$

21. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

22. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log x + \sqrt{a^2 + x^2} + c$

23. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log x + \sqrt{x^2 - a^2} + c$

24. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + c$

25. $\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$

26. $\int \sqrt{x^2 + a^2}\ dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{1}{2}a^2 \log\left|x + \sqrt{x^2 + a^2}\right| + c$

27. $\int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{1}{2}a^2 \log x + \sqrt{x^2 - a^2} + c$

28. $\int e^x\ [f(x) + f'(x)]\, dx = e^x f(x) + c$

29. **Use of Trigonometric Identities in Integration.**

(i) $\sin^2 x = \frac{1 - \cos 2x}{2}$, $\cos^2 x = \frac{1 + \cos 2x}{2}$

(ii) $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$, $\cos^3 x = \frac{3\cos x + \cos 3x}{4}$

(iii) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$
$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$
$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$
$2 \sin A \sin B = \cos(A - B) + \cos(A + B)$

(iv) $\sin x = 2 \sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right)$

30.(i) $1 + 2 + 3 + \cdots\cdots + n = \frac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + \cdots\cdots + n^2 = \frac{n(n+1)(2n-1)}{6}$

(iii) $1^3 + 2^3 + 3^2 + \cdots\cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

(iv) $a + (a + d) + (a + 2d) + \cdots\cdots + [a + (n - 1) d] = \frac{n}{2}\ [2a + (n - 1) d]$

(v) $a + ar + ar^2 + \cdots\cdots + ar^{n+1} = \frac{a(r^n - 1)}{r - 1}$