CBSE Class 12 Maths Notes Chapter 6 Application of Derivatives

Rate of Change of Quantities: Let y = f(x) be a function of x. Then, $\frac{dy}{dx}$ represents the rate of change of y with respect to x. Also, [latex s=1]\frac { dy }{ dx }[/latex]x = x0 represents the rate of change of y with respect to x at x = x_0.

If two variables x and y are varying with respect to another variable t, i.e. x = f(t) and y = g(t), then

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \text{ where } \frac{dx}{dt} \neq 0 \text{ (by chain rule)}$$

In other words, the rate of change of y with respect to x can be calculated using the rate of change of y and that of x both with respect to t.
Note: $\frac{dy}{dx}$ is positive, if y increases as x increases and it is negative, if y decreases as x increases, dx

Marginal Cost: Marginal cost represents the instantaneous rate of change of the total cost at any level of output.
If C(x) represents the cost function for x units produced, then marginal cost (MC) is given by

$$\text{MC} = \frac{d}{dx}\{C(x)\}$$

Marginal Revenue: Marginal revenue represents the rate of change of total revenue with respect to the number of items sold at an instant.
If R(x) is the revenue function for x units sold, then marginal revenue (MR) is given by

$$\text{MR} = \frac{d}{dx}\{R(x)\}$$

Let I be an open interval contained in the domain of a real valued function f. Then, f is said to be

- increasing on I, if $x_1 < x_2$ in I ⇒ $f(x_1) \leq f(x_2)$, ∀ $x_1, x_2 \in I$.
- strictly increasing on I, if $x_1 < x_2$ in I ⇒ $f(x_1) < f(x_2)$, ∀ $x_1, x_2 \in I$.
- decreasing on I, if $x_1 < x_2$ in I ⇒ $f(x_1) \geq f(x_2)$, ∀ $x_1, x_2 \in I$.
- strictly decreasing on I, if $x_1 < x_2$ in $f(x_1) > f(x_2)$, ∀ $x_1, x_2 \in I$.

Let x_0 be a point in the domain of definition of a real-valued function f, then f is said to be increasing, strictly increasing, decreasing or strictly decreasing at x_0, if there exists an open interval I containing x0 such that f is increasing, strictly increasing, decreasing or strictly decreasing, respectively in I.
Note: If for a given interval I ⊆ R, function f increase for some values in I and decrease for other values in I, then we say function is neither increasing nor decreasing.

Let f be continuous on [a, b] and differentiable on the open interval (a, b). Then,

- f is increasing in [a, b] if f'(x) > 0 for each x ∈ (a, b).
- f is decreasing in [a, b] if f'(x) < 0 for each x ∈ (a, b).
- f is a constant function in [a, b], if f'(x) = 0 for each x ∈ (a, b).

Note:
(i) f is strictly increasing in (a, b), if f'(x) > 0 for each x ∈ (a, b).
(ii) f is strictly decreasing in (a, b), if f'(x) < 0 for each x ∈ (a, b).

Monotonic Function: A function which is either increasing or decreasing in a given interval I, is called monotonic function.

Approximation: Let y = f(x) be any function of x. Let Δx be the small change in x and Δy be the corresponding change in y.

i.e. Δy = f(x + Δx) − f(x).Then, dy = f'(x) dx or dy = $\frac{dy}{dx}$ Δx is a good approximation of Δy, when dx = Δx is relatively small and we denote it by dy ~ Δy.
Note:

(i) The differential of the dependent variable is not equal to the increment of the variable whereas the differential of the independent variable is equal to the increment of the variable.
(ii) Absolute Error The change Δx in x is called absolute error in x.

Tangents and Normals
Slope: (i) The slope of a tangent to the curve y = f(x) at the point (x_1, y_1) is given by

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} \text{ or } f'(x_1).$$

(ii) The slope of a normal to the curve y = f(x) at the point (x_1, y_1) is given by

$$\frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$$

Note: If a tangent line to the curve y = f(x) makes an angle θ with X-axis in the positive direction, then $\frac{dy}{dx}$ = Slope of the tangent = tan θ. dx

Equations of Tangent and Normal

The equation of tangent to the curve y = f(x) at the point $P(x_1, y_1)$ is given by
$y - y_1 = m(x - x_1)$, where m = $\frac{dy}{dx}$ at point (x_1, y_1).

The equation of normal to the curve y = f(x) at the point $Q(x_1, y_1)$ is given by
$y - y_1 = \frac{-1}{m}(x - x_1)$, where m = $\frac{dy}{dx}$ at point (x_1, y_1).

If slope of the tangent line is zero, then tanθ = θ, so θ = 0, which means that tangent line is parallel to the X-axis and then equation of tangent at the point (x_1, y_1) is $y = y_1$.

If $\theta \to \frac{\pi}{2}$, then $\tan\theta \to \infty$ which means that tangent line is perpendicular to the X-axis, i.e. parallel to the Y-axis and then equation of the tangent at the point (x_1, y_1) is $x = x_0$.

Maximum and Minimum Value: Let f be a function defined on an interval I. Then,
(i) f is said to have a maximum value in I, if there exists a point c in I such that
f(c) > f(x), ∀ x ∈ I. The number f(c) is called the maximum value of f in I and the point c is called a point of a maximum value of f in I.
(ii) f is said to have a minimum value in I, if there exists a point c in I such that f(c) < f(x), ∀ x ∈ I. The number f(c) is called the minimum value of f in I and the point c is called a point of minimum value of f in I.

(iii) f is said to have an extreme value in I, if there exists a point c in I such that f(c) is either a maximum value or a minimum value of f in I. The number f(c) is called an extreme value off in I and the point c is called an extreme point.

Local Maxima and Local Minima

(i) A function f(x) is said to have a local maximum value at point x = a, if there exists a neighbourhood (a – δ, a + δ) of a such that f(x) < f(a), ∀ x ∈ (a – δ, a + δ), x ≠ a. Here, f(a) is called the local maximum value of f(x) at the point x = a. (ii) A function f(x) is said to have a local minimum value at point x = a, if there exists a neighbourhood (a – δ, a + δ) of a such that f(x) > f(a), ∀ x ∈ (a – δ, a + δ), x ≠ a. Here, f(a) is called the local minimum value of f(x) at x = a.

The points at which a function changes its nature from decreasing to increasing or vice-versa are called turning points.
Note:
(i) Through the graphs, we can even find the maximum/minimum value of a function at a point at which it is not even differentiable.
(ii) Every monotonic function assumes its maximum/minimum value at the endpoints of the domain of definition of the function.

Every continuous function on a closed interval has a maximum and a minimum value.

Let f be a function defined on an open interval I. Suppose cel is any point. If f has local maxima or local minima at x = c, then either f'(c) = 0 or f is not differentiable at c.

Critical Point: A point c in the domain of a function f at which either f'(c) = 0 or f is not differentiable, is called a critical point of f.

First Derivative Test: Let f be a function defined on an open interval I and f be continuous of a critical point c in I. Then,

- if f'(x) changes sign from positive to negative as x increases through c, then c is a point of local maxima.
- if f'(x) changes sign from negative to positive as x increases through c, then c is a point of local minima.
- if f'(x) does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Such a point is called a point of inflection.

Second Derivative Test: Let f(x) be a function defined on an interval I and c ∈ I. Let f be twice differentiable at c. Then,
(i) x = c is a point of local maxima, if f'(c) = 0 and f”(c) < 0. (ii) x = c is a point of local minima, if f'(c) = 0 and f”(c) > 0.
(iii) the test fails, if f'(c) = 0 and f”(c) = 0.

Note
(i) If the test fails, then we go back to the first derivative test and find whether a is a point of local maxima, local minima or a point of inflexion.
(ii) If we say that f is twice differentiable at o, then it means second order derivative exists at a.

Absolute Maximum Value: Let f(x) be a function defined in its domain say $Z \subset R$. Then, f(x) is said to have the maximum value at a point $a \in Z$, if $f(x) \leq f(a), \forall x \in Z$.

Absolute Minimum Value: Let f(x) be a function defined in its domain say $Z \subset R$. Then, f(x) is said to have the minimum value at a point $a \in Z$, if $f(x) \geq f(a), \forall x \in Z$.

Note: Every continuous function defined in a closed interval has a maximum or a minimum value which lies either at the end points or at the solution of $f'(x) = 0$ or at the point, where the function is not differentiable.

Let f be a continuous function on an interval I = [a, b]. Then, f has the absolute maximum value and/attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.