CBSE Class 12 Maths Notes Chapter 4 Determinants

Determinant: Determinant is the numerical value of the square matrix. So, to every square matrix A = $[a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A. It is denoted by det A or |A|.
Note
(i) Read |A| as determinant A not absolute value of A.
(ii) Determinant gives numerical value but matrix do not give numerical value.
(iii) A determinant always has an equal number of rows and columns, i.e. only square matrix have determinants.

Value of a Determinant

Value of determinant of a matrix of order 2, A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

Value of determinant of a matrix of order 3, A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is given by expressing it in terms of second order determinant. This is known as expansion of a determinant along a row (or column).

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \text{(expansion along first row } R_1\text{)}$$

Note
(i) For easier calculations of determinant, we shall expand the determinant along that row or column which contains the maximum number of zeroes.
(ii) While expanding, instead of multiplying by $(-1)^{i+j}$, we can multiply by +1 or -1 according to as (i + j) is even or odd.

Let A be a matrix of order n and let |A| = x. Then, $|kA| = k^n |A| = k^n x$, where n = 1, 2, 3,...

Minor: Minor of an element ay of a determinant, is a determinant obtained by deleting the ith row and jth column in which element ay lies. Minor of an element a_{ij} is denoted by M_{ij}.
Note: Minor of an element of a determinant of order n(n ≥ 2) is a determinant of order (n – 1).

Cofactor: Cofactor of an element a_{ij} of a determinant, denoted by A_{ij} or C_{ij} is defined as $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is a minor of an element a_{ij}.
Note
(i) For expanding the determinant, we can use minors and cofactors as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} A_{11} - a_{12} A_{12} + a_{13} A_{13}$$

(ii) If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

Singular and non-singular Matrix: If the value of determinant corresponding to a square matrix is zero, then the matrix is said to be a singular matrix, otherwise it is non-singular matrix, i.e. for a square matrix A, if $|A| \neq 0$, then it is said to be a non-singular matrix and of $|A| = 0$, then it is said to be a singular matrix.
Theorems
(i) If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.
(ii) The determinant of the product of matrices is equal to the product of their respective determinants, i.e. $|AB| = |A||B|$, where A and B are a square matrix of the same order.

Adjoint of a Matrix: The adjoint of a square matrix 'A' is the transpose of the matrix which obtained by cofactors of each element of a determinant corresponding to that given matrix. It is denoted by adj(A).
In general, adjoint of a matrix $A = [a_{ij}]_{n\times n}$ is a matrix $[A_{ji}]_{n\times n}$, where A_{ji} is a cofactor of element a_{ji}.

Properties of Adjoint of a Matrix
If A is a square matrix of order n × n, then

- $A(\text{adj } A) = (\text{adj } A)A = |A| I_n$
- $|\text{adj } A| = |A|^{n-1}$
- $\text{adj } (A^T) = (\text{adj } A)^T$

The area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

NOTE: Since the area is a positive quantity we always take the absolute value of the determinant.

Properties of Determinants

To find the value of the determinant, we try to make the maximum possible zero in a row (or a column) by using properties given below and then expand the determinant corresponding that row (or column).

Following are the various properties of determinants:

1. If all the elements of any row or column of a determinant are zero, then the value of a determinant is zero.

2. If each element of any one row or one column of a determinant is a multiple of scalar k, then the value of the determinant is a multiple of k. then the value of the determinant is a multiple of k. i.e.

$$\begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix} = k\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

3. If in a determinant any two rows or columns are interchanged, then the value of the determinant obtained is negative of the value of the given determinant. If we make n such changes of rows (columns) indeterminant Δ and obtain determinant Δ , then Δ_1 = $(-1)^n$ Δ.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$$

4. If all corresponding elements of any two rows or columns of a determinant are identical or proportional, then the value of the determinant is zero.

$$\begin{vmatrix} a & b & c \\ b & e & f \\ a & b & c \end{vmatrix} = 0$$

[∴ R_1 and R_3 are identical.]

5. The value of a determinant remains unchanged on changing rows into columns and columns into rows. It follows that, if A is a square matrix, then |A'| = |A|.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

Note: det(A) = det(A'), where A' = transpose of A.

6. If some or all elements of a row or column of a determinant are expressed as a sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants, i.e.

$$\begin{vmatrix} a+a' & b+b' & c+c' \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix}.$$

7. In the elements of any row or column of a determinant, if we add or subtract the multiples of corresponding elements of any other row or column, then the value of determinant remains unchanged, i.e.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+kb & b & c \\ d+ke & e & f \\ g+kh & h & i \end{vmatrix} (C_1 \to C_1 + kC_2)$$

In other words, the value of determinants remains the same, if we apply the operation $R_i \to R_i + kE_j$ or $C_i \to C_j \to kC_j$.

Inverse of a Matrix and Applications of Determinants and Matrix

1. Inverse of a Square Matrix: If A is a non-singular matrix (i.e. |A| ≠ 0), then

$$A^{-1} = \frac{1}{|A|}\text{adj}(A)$$

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the inverse is $A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, where A_{ij} is the cofactor of A.

Note: Inverse of a matrix, if exists, is unique.

Properties of a Inverse Matrix

- $(A^{-1})^{-1} = A$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
- $\text{adj}\,(A^{-1}) = (\text{adj}\,A)^{-1}$

2. Solution of system of linear equations using inverse of a matrix.

Let the given system of equations be $a_1x + b_1y + c_1z = d_1$; $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$.
We write the following system of linear equations in matrix form as AX = B, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

Case I: If $|A| \neq 0$, then the system is consistent and has a unique solution which is given by $X = A^{-1}B$.
Case II: If $|A| = 0$ and $(\text{adj } A)\, B \neq 0$, then system is inconsistent and has no solution.
Case III: If $|A| = 0$ and $(\text{adj } A)\, B = 0$, then system may be either consistent or inconsistent according to as the system have either infinitely many solutions or no solutions