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PRACTICE PAPER 06 (2023-24)

CHAPTER 06 APPLICATION OF DERIVATIVES

(ANSWERS)

SUBJECT: MATHEMATICS **MAX. MARKS : 40**

CLASS : XII **DURATION : 1½ hrs**

General Instructions:

(i). **All** questions are compulsory.

(ii). This question paper contains 2**0** questions divided into five Sections A, B, C, D and E.

(iii). **Section A** comprises of **10 MCQs** of **1 mark** each. **Section B** comprises of 4 questions of **2 marks** each. **Section C** comprises of 3 questions of **3 marks** each. **Section D** comprises of 1 question of **5 marks** each and **Section E** comprises of 2 Case Study Based Questions of **4 marks** each.

(iv). There is no overall choice.

(v). Use of Calculators is not permitted

SECTION – A

Questions 1 to 10 carry 1 mark each.

1. The function $f(x) = 2x^3 – 15x^2 + 36x + 6$ is increasing in the interval

(a) $(-\infty, 2) \cup (3, \infty)$ (b) $(-\infty, 2)$ (c) $(-\infty, 2] \cup [3, \infty)$ (d) $[3, \infty)$

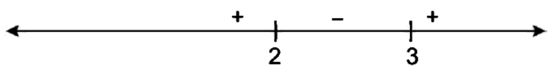
Ans: (c) $(-\infty, 2] \cup [3, \infty)$

Given $f(x) = 2x^3 - 15x^2 + 36x + 6$

$$f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$$

$$= 6(x^2 - 2x - 3x + 6) = 6\{x(x-2) - 3(x-2)\}$$

$$= 6(x-2)(x-3)$$



$\because \quad f'(x) > 0$ if $x \in (-\infty, 2] \cup [3, \infty)$

2. The maximum value of $\left(\frac{1}{x}\right)^x$ is

(a) e (b) e^e (c) $e^{1/e}$ (d) $\left(\frac{1}{e}\right)^{1/e}$

Ans:

Let $y = \left(\frac{1}{x}\right)^x \Rightarrow \log y = x \cdot \log\frac{1}{x}$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right) + \log\frac{1}{x}\cdot 1 = -1 + \log\frac{1}{x} \qquad \text{[Differentiate both sides]}$$

$$\Rightarrow \frac{dy}{dx} = \left(\log\frac{1}{x} - 1\right)\cdot\left(\frac{1}{x}\right)^x$$

Now, $\frac{dy}{dx} = 0 \Rightarrow \log\frac{1}{x} = 1 = \log e \Rightarrow \frac{1}{x} = e$

$\therefore \quad x = \frac{1}{e}$

Hence, the maximum value of $f\left(\frac{1}{e}\right) = (e)^{1/e}$.

3. A ladder, 5 meter long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 metres from the wall is
(a) 1/10 radian/sec (b) 1/20 radian/sec (c) 20 radian/sec (d) 10 radian/sec
Ans: (b) 1/20 radian/sec

4. If f(x) = a(x – cos x) is strictly decreasing in R, then 'a' belongs to
(a) {0} (b) (0, ∞) (c) (-∞, 0) (d) (-∞, ∞)
Ans: (c) (-∞, 0)

Given function, $f(x) = a(x - \cos x) \Rightarrow f'(x) = a(1 + \sin x)$

For $f(x)$ to be decreasing

$\Rightarrow f'(x) < 0 \Rightarrow a(1 + \sin x) < 0$

$\Rightarrow a < 0 \quad (-1 \le \sin \le 1 \Rightarrow 0 \le 1 + \sin x \le 2)$

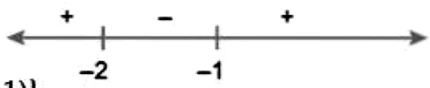
$\Rightarrow a \in (-\infty, 0)$

5. The interval in which the function f(x) = $2x^3 + 9x^2 + 12x - 1$ is decreasing is
(a) (-1, ∞) (b) (-2,-1) (c) (-∞, -2) (d) [-1,1]
Ans:

$f(x) = 2x^3 + 9x^2 + 12x - 1$

$f'(x) = 6x^2 + 18x + 12 = 6(x^2 + 3x + 2)$

$= 6(x^2 + x + 2x + 2) = 6\{x(x + 1) + 2(x + 1)\}$



$\Rightarrow \quad f'(x) = 6(x + 1)(x + 2)$

$\because \quad f'(x) < 0 \ \forall \ x \in (-2, -1)$

6. The value of b for which the function f(x) = x + cos x + b is strictly decreasing over R is
(a) b < 1 (b) No value of b exists (c) b ≤ 1 (d) b ≥ 1
Ans:

$f'(x) = 1 - \sin x, f'(x) \ge 0 \ \forall \ x \in \mathbb{R}$

$\Rightarrow$ No value of b exists.

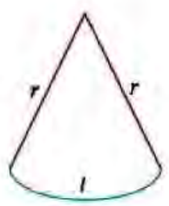
7. A wire of length 20 cm is bent in the form of a sector of a circle. The maximum area that can be enclosed by the wire is:
(a) 20 sq cm (b) 25 sq cm (c) 10 sq cm (d) 30 sq cm
Ans: (b) 25 sq cm

Let r be the radius of circle and l be the arc length of the sector of the circle.

$\therefore$ Perimeter of the sector = $2r + l$

$20 = 2r + l \Rightarrow l = 20 - 2r$

Now, area of sector, $A = \frac{1}{2} lr$



$\Rightarrow A = \frac{1}{2} \times (20 - 2r) \times r \quad \Rightarrow A = 10r - r^2$

For area to be maximum or minimum we have

$\frac{dA}{dr} = 0 \quad \Rightarrow 10 - 2r = 0 \quad \Rightarrow r = 5$

Now, $\frac{d^2A}{dr^2} = -2 < 0$

$\Rightarrow$ Area will be maximum at $r = 5$

$\Rightarrow$ Area = $10 \times 5 - (5)^2 = 50 - 25 = 25$ sq. cm

8. The value of x for which $(x - x^2)$ is maximum, is:

(a) 3/4 (b) 1/2 (c) 1/3 (d) 1/4

Ans:

Let $y = x - x^2$

For y to be maximum or minimum.

$$\frac{dy}{dx} = 0 \Rightarrow \frac{d(x - x^2)}{dx} = 0 \Rightarrow 1 - 2x = 0 \Rightarrow x = \frac{1}{2}$$

Now, $\frac{d^2y}{dx^2} = -2 < 0 \Rightarrow$ y will be maximum at $x = \frac{1}{2}$.

In the following questions 9 and 10, a statement of assertion (A) is followed by a statement of reason (R). Mark the correct choice as:

(a) Both Assertion (A) and Reason (R) are true and Reason(R) is the correct explanation of assertion (A).

(b) Both Assertion (A) and Reason (R) are true but Reason(R) is not the correct explanation of assertion (A).

(c) Assertion (A) is true but reason (R) is false.

(d) Assertion (A) is false but reason (R) is true.

9. **Assertion (A) :** $f(x) = x^4$ is decreasing in the interval $(0, \infty)$.

Reason (R) : Any function $y = f(x)$ is decreasing if $\frac{dy}{dx} < 0$

Ans: (d) Assertion (A) is false but reason (R) is true.

$f(x) = x^4 \Rightarrow f'(x) = 4x^3$

For decreasing function

$$f'(x) < 0 \Rightarrow 4x^3 < 0$$

$$\Rightarrow x^3 < 0 \Rightarrow x < 0$$

$$\Rightarrow x \in (-\infty, 0).$$

Clearly, Assertion (A) is false and Reason (R) is true.

10. **Assertion (A):** The rate of change of area of a circle with respect to its radius r when r = 6 cm is $12\pi cm^2/cm$.

Reason (R): Rate of change of area of a circle with respect to its radius r is $\frac{dA}{dr}$, where A is the area of the circle.

Ans: (a) Both assertion (A) and reason (R) are true & Reason (R) is the correct explanation of assertion (A).

We have, $\frac{dA}{dr} = \frac{d\pi r^2}{dr} = \pi \times 2r = 2\pi r$

$$\left.\frac{dA}{dr}\right|_{r=6} = 2\pi \times 6 = 12\pi \text{ cm}^2/\text{cm}$$

Clearly, both Assertion(A) and Reason(R) are true and Reason(R) is the correct explanation of Assertion(A).

SECTION – B

Questions 11 to 14 carry 2 marks each.

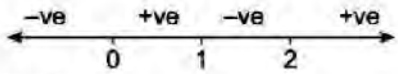
11. Find the values of x for which $y = [x(x - 2)]^2$ is an increasing function.

Ans:

Given, $y = [x(x-2)]^2$

$\therefore \quad \frac{dy}{dx} = 2[x(x-2)] \times (2x-2) = 4x(x-1)(x-2)$

For increasing function, $\frac{dy}{dx} > 0$



$4x(x-1)(x-2) > 0 \quad \Rightarrow \quad x(x-1)(x-2) > 0$

From sign rule,

For $\frac{dy}{dx} > 0$ value of $x = 0 < x < 1$ and $x > 2$

Therefore, y is increasing $\forall\, x \in (0, 1) \cup (2, \infty)$.

12. Show that the function f defined by $f(x) = (x - 1)e^x + 1$ is an increasing function for all $x > 0$.
Ans: $f(x) = (x - 1)e^x + 1$

$\Rightarrow f'(x) = xe^x$

Now $x > 0$ and $e^x > 0$ for all x

$\therefore f'(x) > 0 =$ f is increasing function.

13. Find the rate of change of volume of sphere with respect to its surface area, when radius is 2 cm.
Ans: Let r be the radius of sphere, V be the volume and S be the surface area of sphere.

$\because \quad V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$

$\because \quad \frac{dV}{dr} = 4\pi r^2$ and $\frac{dS}{dr} = 8\pi r$

$\therefore \quad \frac{dV}{dS} = \frac{4\pi r^2}{8\pi r} = \frac{1}{2}r$

$\therefore \quad \left.\frac{dV}{dS}\right|_{r = 2cm} = \frac{2}{2} = 1\,\text{cm}^3/1\text{cm}^2$

14. The amount of pollution content added in air in a city due to x-diesel vehicles is given by $P(x) = 0.005x^3 + 0.02x^2 + 30x$. Find the marginal increase in pollution content when 3 diesel vehicles are added.
Ans: We have to find $[P'(x)]_{x=3}$

Now, $P(x) = 0.005x^3 + 0.02x^2 + 30x$

$\therefore P'(x) = 0.015x^2 + 0.04x + 30$

$\Rightarrow [P'(x)]_{x=3} = 0.015 \times 9 + 0.04 \times 3 + 30$

$= 0.135 + 0.12 + 30 = 30.255$

SECTION – C

Questions 15 to 17 carry 3 marks each.

15. Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is

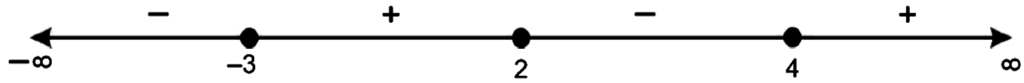
(a) strictly increasing (b) strictly decreasing.
Ans:

$f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12 \Rightarrow f'(x) = \frac{4x^3}{4} - 3x^2 - 5 \times 2x + 24 \times 1 + 0$

$f'(x) = x^3 - 3x^2 - 10x + 24$ [Since $x = 2$ is a factor, using Hit and trial error method factorise]

$f'(x) = (x-2)(x-4)(x+3) \Rightarrow f'(x) = 0$

$\Rightarrow x = 2, 4, -3$



$\therefore$ $f(x)$ is strictly increasing on $(-3, 2) \cup (4, \infty)$ and $f(x)$ is strictly decreasing on $(-\infty, -3) \cup (2, 4)$.

16. Find the absolute maximum and absolute minimum values of the function f given by $f(x) = \sin^2 x - \cos x, x \in [0, \pi]$.

Ans:

Here, $f(x) = \sin^2 x - \cos x$

$f'(x) = 2\sin x.\cos x + \sin x \Rightarrow f'(x) = \sin x(2\cos x + 1)$

For critical point: $f'(x) = 0$

$\Rightarrow \sin x(2\cos x + 1) = 0 \Rightarrow \sin x = 0$ or $\cos x = -\frac{1}{2}$

$\Rightarrow x = 0$ or $\cos x = \cos\frac{2\pi}{3} \Rightarrow x = 0$ or $x = 2n\pi \pm \frac{2\pi}{3}$, where $n = 0, \pm 1, \pm 2 \ldots$

$\Rightarrow x = 0$ or $x = \frac{2\pi}{2}$ other values does not belong to $[0, \pi]$.

For absolute maximum or minimum values:

$f(0) = \sin^2 0 - \cos 0 = 0 - 1 = -1$

$f\left(\frac{2\pi}{3}\right) = \sin^2\frac{2\pi}{3} - \cos\frac{2\pi}{3} = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

$f(\pi) = \sin^2 \pi - \cos \pi = 0 - (-1) = 1$

Hence, absolute maximum value $= \frac{5}{4}$ and absolute minimum value $= -1$.

17. The volume of a cube is increasing at the rate of 9 cm3/s. How fast is its surface area increasing when the length of an edge is 10 cm?

Ans:

Let V and S be the volume and surface area of a cube of side x cm respectively.

Given $\frac{dV}{dt} = 9 \text{ cm}^3/\text{sec}$

We require $\left.\frac{dS}{dt}\right]_{x = 10 \text{ cm}}$

Now $V = x^3$

$\Rightarrow \frac{dV}{dt} = 3x^2.\frac{dx}{dt} \Rightarrow 9 = 3x^2.\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{9}{3x^2} = \frac{3}{x^2}$

Again, $\because S = 6x^2$ [By formula for surface area of a cube]

$\Rightarrow \frac{dS}{dt} = 12.x.\frac{dx}{dt} = 12x.\frac{3}{x^2} = \frac{36}{x}$

$\Rightarrow \left.\frac{dS}{dt}\right]_{x = 10 \text{ cm}} = \frac{36}{10} = 3.6 \text{ cm}^2/\text{sec}.$

SECTION – D

Questions 18 carry 5 marks.

18. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

Ans:

Let x be radius and $(y + R)$ be the height of cylinder given radius of sphere be R.

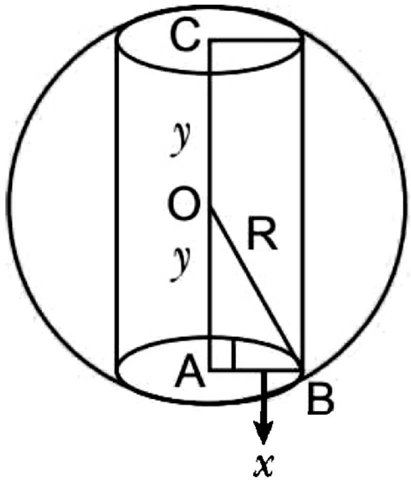
In ΔOAB, we have,

$$OB^2 = OA^2 + AB^2$$

$$\Rightarrow \quad R^2 = y^2 + x^2 \quad \Rightarrow \quad x^2 + y^2 = R^2 \quad \Rightarrow \quad x^2 = R^2 - y^2 \quad ...(i)$$

Now, volume of cylinder = $\pi x^2 \times 2y$

$$\Rightarrow \quad V = \pi(R^2 - y^2) \times 2y$$



For volume to be maximum or minimum

$$\frac{dV}{dy} = 0 \quad \Rightarrow \quad 2\pi\{(R^2 - y^2) \times 1 + y \times (-2y)\} = 0$$

$$\Rightarrow \quad R^2 - y^2 - 2y^2 = 0 \quad \Rightarrow \quad R^2 - 3y^2 = 0 \quad \Rightarrow \quad R^2 = 3y^2$$

$$\Rightarrow \quad y^2 = \frac{R^2}{3} \quad \Rightarrow \quad y = \frac{R}{\sqrt{3}}$$

$$\because \quad \frac{dV}{dy} = 2\pi(R^2 - 3y^2)$$

$$\therefore \quad \frac{d^2V}{dy^2}\Big|_{\left(\text{at } y = \frac{R}{\sqrt{3}}\right)} = 2\pi(-6y) = -12\pi y = \frac{-12R\pi}{\sqrt{3}} < 0$$

$\therefore$ Volume will be maximum when $y = \frac{R}{\sqrt{3}}$

$\therefore$ Height of cylinder = $2y = \frac{2R}{\sqrt{3}}$

and maximum volume $= \pi(R^2 - y^2) \times 2y$

$$= \pi\left(R^2 - \frac{R^2}{3}\right) \times \frac{2R}{\sqrt{3}} = \pi \times \frac{2R^2}{3} \times \frac{2R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}$$

SECTION – E (Case Study Based Questions)

Questions 19 to 20 carry 4 marks each.

19. Case-Study 1: An owner of a car rental company have determined that if they charge customers Rs x per day to rent a car, where $50 \le x \le 200$, then number of cars (n), they rent per day can be shown by linear function $n(x) = 1000 - 5x$. If they charge Rs. 50 per day or less they will rent all their cars. If they charge Rs. 200 or more per day they will not rent any car.

Based on the above information, answer the following question.

(i) If R(x) denote the revenue, then find the value of x at which R(x) has maximum value.
(ii) Find the Maximum revenue collected by company

OR

Find the number of cars rented per day, if x = 75.

Ans: (i) Let x be the price charge per car per day and n be the number of cars rented per day.

$R(x) = n \times x = (1000 - 5x)\, x = -5x^2 + 1000x$

$\Rightarrow R'(x) = 1000 - 10x$

For R(x) to be maximum or minimum, $R'(x) = 0$

$\Rightarrow -10x + 1000 = 0 \Rightarrow x = 100$

Also, $R''(x) = -10 < 0$

Thus, R(x) is maximum at x = 100

(ii) At x = 100, R(x) is maximum.

Maximum revenue = R(100) = –5(100)2 + 1000(100) = Rs. 50,000

OR

If x = 75, number of cars rented per day is given by

$n = 1000 - 5 \times 75 = 625$

20. Case-Study 2: Sonam wants to prepare a sweet box for Diwali at home. For making lower part of box, she takes a square piece of cardboard of side 18 cm.

Now, x cm be the length of each side of the square cardboard which is to be cut off from corner of the square piece of side 18 cm.

Based on the above information, answer the following questions :

(i) Express Volume of the open box formed by folding up the cutting corner in terms of x and find the value of x for which $\frac{dV}{dx} = 0$.

(ii) Sonam is interested in maximising the volume of the box. So, what should be the side of the square to be cut off so that the volume of the box is maximum?

Ans: (i) height of open box = x cm

Length of open box = 18 – 2x

and width of open box = 18 – 2x

$\therefore$ Volume (V) of the open box = $x \times (18 - 2x) \times (18 - 2x)$

$\Rightarrow V = x(18 - 2x)^2$

$\Rightarrow \frac{dV}{dx} = x \cdot 2(18 - 2x)(-2) + (18 - 2x)^2$

$= (18 - 2x)(-4x + 18 - 2x)$

$= (18 - 2x)(18 - 6x)$

Now, $\frac{dV}{dx} = 0 \Rightarrow 18 - 2x = 0$ or $18 - 6x = 0$

$\Rightarrow x = 9$ or 3

(ii) We have, V = $x(18 - 2x)^2$ and $\frac{dV}{dx} = (18 - 2x)(18 - 6x)$

$\Rightarrow \frac{d^2V}{dx^2} = (18 - 2x)(-6) + (18 - 6x)(-2)$

$= (-2)[54 - 6x + 18 - 6x]$

$= (-2)[72 - 12x] = 24x - 144$

For x = 3, $\frac{d^2V}{dx^2} < 0$

and for x = 9, $\frac{d^2V}{dx^2} > 0$

So, volume will be maximum when x = 3.